UNIVERSITY OF PISA

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Thesis

STRICHARTZ ESTIMATES FOR NONLINEAR WAVE EQUATION

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0.1 Introduction

One of the most studied mathematical models in the modern and contemporary analysis area is represented by the Cauchy problem associated to the nonlinear wave equation (NLW)

$$\begin{cases} (-\partial_t^2 + \Delta_{\mathbb{R}^n})u = F(u, u_t, Du) \\ u(0, x) = g(x) \\ \partial_t u(0, x) = h(x) \end{cases}$$

where the initial conditions g, h are functions (distributions) whose regularity is set. This equation is meant to model phenomena of every nature (like acoustics, electromagnetism, quantum mechanics, fluid dynamics) and all of them have in common the way in which the information (not necessarily material propagation) spreads away: the wave.

The first obstacle to the study of this equation is no doubt establishing conditions of existence of a solution - at least a time-local one - and its regularty, which is important too. In fact, while we're able to say a lot of things about the linear equation, NLW still today conceals secrets and several approaches are being tempted in order to learn about it.

As we'll see in Chapter 2, classical subject of hyperbolic PDE theory, like energy estimates or contraction theorem, will allow us to give an answer to the previous question and obtain a local uniqueness given regular enough inital data and nonlinearity. More precisely, we will state the existence of such a solution when considering $F \in C^\infty$ with $F(\underline{0}) = 0$, $g \in H^k(\mathbb{R}^n)$ and $h \in H^{k-1}(\mathbb{R}^n)$ as $k > \frac{n}{2} + 1$.

Several mathematicians (Sogge, Ponce, Sideris) focused their work on particular nonlinearities in well set dimensions and obtained fairly good results by proceeding in this direction. However, the classical analysis is struggling in giving some new ideas and methods and a new discipline, the harmonic analysis, is fastly arising and eager to prove its validity in a still little-known field: the real revolution in this field is represented by the introduction of specific solution decay estimates, due to American mathematician Robert Stephen Strichartz, which permitted to reach far better results. These estimates generally enhance (of about one order) the loss of derivatives about regularity of the solution, by the way there are some particular situations - some examples are given in Chapter 4 - in which the classical approach gives worse information and sometime fails; nevertheless, this new argument is extremely elegant and efficient compared with the previous one.

Another matter of great interest is speaking about time-global existence. It's well-known that this issue has a negative answer and one can hope to obtain it only by conveniently performing on nonlinearity and/or largeness of the initial data (in some suitable norm). As the most famous representative equation of the former, we recall

$$(-\partial_t^2 + \Delta_{\mathbb{R}^n})u = u|u|^{p-1}$$

and thanks to classical methods one can show that, given suitable conditions of regularity in inital data, the local solution can be extended to a global one if and only if $1 \leq p \leq 5$. Also, we'll show how Strichartz estimates make the proof in the critical nonlinearity case $p = 5$ much easier. Further information can be read in [9], [10], [11], [21], [28]. The latter is made of situations like

$$\begin{cases} (-\partial_t^2 + \Delta_{\mathbb{R}^n})u = F(u, u_t, Du) \\ u(0, x) = \epsilon g(x) \\ \partial_t u(0, x) = \epsilon h(x) \end{cases}$$

and that has been examined in depth by Lindblad e Sogge; again, suitable requests on nonlinearity allow us to find a global solution to the problem.

Historically speaking, Strichartz inequalities were born in 1977 in the article *Restriction of Fourier Transform to Quadratic Surfaces and Decay of Solutions of Wave Equations*, presented, conceived and proved in L^p spaces. Later works, with the help of Lindblad, Sogge, Ginibre, Velo, Tao and others, made these estimate cleaner and were inserted in more general and suitable settings, Besov spaces

$$\dot{B}^p_{r,s} = \{u \in S' : ||u||_{\dot{B}^p_{r,s}} = ||2^{pj}\phi_j * u||_{l^s_j L^r_x} < \infty\}$$

here, the formulation of the inequalities is more natural, the proof is more readable and only at the end of the process a L^p reading is presented. We will go exactly through this way, by conveniently introducing these spaces and by relating them with Sobolev spaces through some embedding results.

The proof of these inequalities will require non trivial analytic tools and often the introduction of some more concepts, maybe less usual (like fractional derivative), maybe more elegant (TT^* method) will be useful. When the form was to becoming heavy and the abundance of details risked to cover the original idea, we preferred to call directly our sources with their rigorous work.

One of the most thorny aspects of Strichartz estimates is that they radically change when we consider other equations (like NLS): very often, we could not recycle the deep discoveries of these estimate in the

Schrödinger equation field (like in [11]), even if some authors often suggested very original ideas in order to rewrite them in a wave context (see [16]).

In conclusion, by the will to be comprehensive enough, we recall in the appendix the Penrose transform as an example of a different strategy to engage the problem (many authors like Christodolou followed this one), maybe more devoted to a geometric approach of the situation, like the topology or the metric of the spaces involved, in order to solve the (even global) existence issue throuh a compactification argument in the well-known Einstein-Penrose diamond, and then by using the theory of existence of solution of PDE in compact spaces, which is a more traditional theme.

Let's shortly summarize issues and themes we'll talk about in this work:

- In Chapter 1 we will introduce every concepts are needed to state and prove Strichartz inequalities in their Besov formulation; obviously, we will define Besov spaces and their relation with Sobolev spaces; then, we will define fractional derivative through Fourier transform, which we will use as generalization for extending those results mainly stated for integer exponents; we will conclude then with a functional analysis subject known as TT^* lemma, which is crucial in the proof of the inequalities;

- In Chapter 2 we will show existence and uniqueness of a weak solution (conveniently defined) of the problem under certain conditions through a contraption argument, energy estimate and Gronwall lemma;

- In Chapter 3 we will finally show Strichartz estimates and prove them in the most generality by using the tools of the first chapter;

- In Chapter 4 we will restate the results of the previous chapter in their L^p reading and make a concrete example of application by analyzing the critical case of global existence of the solution of

$$\begin{cases} (-\partial_t^2 + \Delta_{\mathbb{R}^n})u = u^5 \\ u(0,x) = g(x) \\ \partial_t u(0,x) = h(x) \end{cases}$$

  studied by Grillakis with very polish and difficult techniques, some of these are described in [9] and deeply analyzed in [26].

Contents

Chapter 1

Introductory concepts

1.1 Besov Spaces and Lisorkin Spaces

As disclosed before, we're going to show some results of existence and uniqueness that we're going to gain through two different roads. The first one is the classical way of proceeding by introducing the right normed spaces and working on them with the contraption theorem. The second one, the newer one, uses some strong inequalities and solution decay estimates. But this last one requires more polished concepts and tools and we immediately begin to present them.

In the spaces $L^r = L^r(\mathbb{R}^n)$ we can fastly characterize the indices r by using the notation $\alpha(r) = \frac{1}{2} - \frac{1}{r} = \frac{r-2}{2r}$ (for $r \leq 2$). We should notice that $\alpha(r)$ is an increasing function of r, $\alpha(r) = 0 \Leftrightarrow r = 2$, all of them are moreover linear in $\frac{1}{r}$ (and then they behave linearly under interpolation).

In the presentation of the results in Chapter 3, the following quantities are convenient:

$$\beta(r) = \frac{n-1}{2}\alpha(r); \quad \gamma(r) = (n-1)\alpha(r); \quad \delta(r) = n\alpha(r)$$

and we notice that for $n \geq 3$ these three values are simultaneously in alphabetic and increasing order. These definition are not casual: as we'll see later in the work, $\beta(r)$ is the loss of derivative of certain estimates (which will be reported later in the suitable functional spaces; $\gamma(r)$ is the exponent of the optimal decay time of the L^r-regular solution of the wave equation; finally, $\delta(r)$, which is the more familiar, regularly appears in Sobolev and Hölder inequalities (in effect, $\frac{n}{r}$ is the degree in x of the $L^r(\mathbb{R}^n)$ norm).

As usual, $\hat{u}$ will denote the Fourier transform of u (that is to say, $Fu = \hat{u}$), while $*_x, *_t$ will respectively denote the space and the time convolution. Let's introduce a particularly clever construction (which is very similar to a partition of unity over $\mathbb{R}^n$), historically due to Paley and Littlewood:

Definition 1.1. *Let $\hat{\psi} \in C_0^\infty(\mathbb{R}^n)$ with $0 \leq \hat{\psi} \leq 1$, $\hat{\psi}(\xi) = 1$ for $|\xi| \leq 1$ and $\hat{\psi}(\xi) = 0$ for $|\xi| \geq 2$. We define $\hat{\phi}_0(\xi) = \hat{\psi}(\xi) - \hat{\psi}(2\xi)$ and*

$\hat{\phi}_j(\xi) = \hat{\phi}_0(2^{-j}\xi)$. *We say that the family* $\hat{\phi}_j(\xi)_{j\in\mathbb{Z}}$ *here defined, plus the condition*

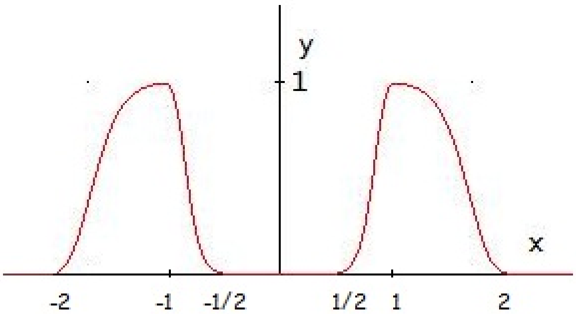
$$\sum_{j\in\mathbb{Z}} \hat{\phi}_j(\xi) = 1 \quad \forall\, \xi \in \mathbb{R}^n - \{0\}$$

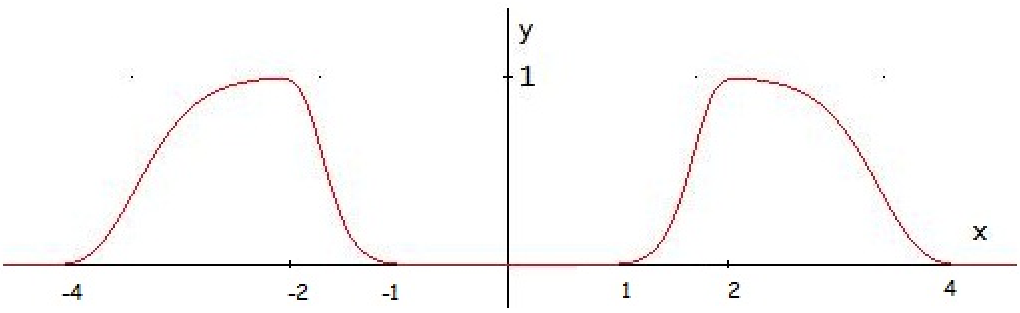
is the ***diadic decomposition*** *of* $\mathbb{R}^n$.

Specifically, by analyzing $\hat{\phi}_j$, we discover that its support is contained in the annuluses

$$Supp\ \hat{\phi}_j \subset \{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$$

moreover $\hat{\phi}_j(0) = 0 \forall\ j$ (here's why we forgot about 0 from the extra condition) and $\hat{\phi}_j(\xi) = 1 \forall\ \xi, |\xi| = 2^j$ (so $\hat{\phi}_j$ are 1 only on the spheres of radius 2^j). We notice that, in this way, the extra condition makes it a good definition since in every point at most two elements of that sum are not 0.





(The images describe a possible representation of $\hat{\phi}_0$ and $\hat{\phi}_1$)

Later in this work we'll need some useful trick:

Lemma 1.2. *Calling* $\tilde{\phi}_j = \phi_{j-1} + \phi_j + \phi_{j+1}$, *we have*

$$\phi_j * u = \tilde{\phi}_j * \phi_j * u \quad \forall u \in S(\mathbb{R}^n).$$

Proof. Since the Fourier transform is linear, we have

$$\hat{\tilde{\phi}}_j = \hat{\phi}_{j-1} + \hat{\phi}_j + \hat{\phi}_{j+1}.$$

Now, it's sufficient to notice that, when $2^{j-1} \leq |\xi| \leq 2^j$ only $\hat{\phi}_{j-1}+\hat{\phi}_j$ are not 0. So, using the extra condition,

$$(\hat{\phi}_{j-1} + \hat{\phi}_j)_{|\{\xi, 2^{j-1} \leq |\xi| \leq 2^j\}} = 1.$$

Then,

$$\hat{\tilde{\phi}}_j \hat{\phi}_j = (\hat{\phi}_{j-1} + \hat{\phi}_j + \hat{\phi}_{j+1})\hat{\phi}_j = \hat{\phi}_j$$

and so, by using a random tempered distribution u,

$$\hat{\tilde{\phi}}_j \hat{\phi}_j \hat{u} = \hat{\phi}_j \hat{u}.$$

Thesis follows from theorem of convolution under the integral sign

$$\phi_j * u = F^{-1}(\hat{\phi}_j \hat{u}) = F^{-1}(\hat{\tilde{\phi}}_j \hat{\phi}_j \hat{u}) = \tilde{\phi}_j * \phi_j * u.$$

□

We shall introduce now the concepts of Besov and Lisorkin space. The deal is that Strichartz estimates have a proof which is very natural and clear if read in these spaces and, morover, they can be reconducted to Sobolev spaces in an easy way thanks to some embedding theorems. We can associate to each (tempered) distribution u the sequence $\{\phi_j * u\}_j$ (so we have C^∞-regular functions) and we will consider it as functions of variables j, x.

Definition 1.3. *The* ***homogeneous Besov space*** *is defined* $\forall\ p, r, s \in \mathbb{R}, r, s \geq 1$ *as*

$$\dot{B}^p_{r,s} = \{u \in S' : ||u||_{\dot{B}^p_{r,s}} = ||2^{pj}\phi_j * u||_{l^s_j L^r_x} < \infty\}$$

where one must first calculate the L^r norm in x and then the l^s norm in j, so reading that as a sequence.

In a perfect analogous way, one can exchange the order of calculating the norms and obtain

$$\dot{F}^p_{r,s} = \{u \in S' : ||u||_{\dot{B}^p_{r,s}} = ||2^{pj}\phi_j * u||_{L^r_x l^s_j} < \infty\}$$

and this last one is called ***homogeneous Triebel-Lizorkin space****.*

In particular, we can write the norms of the two spaces in the following way:

$$||u||_{\dot{B}^p_{r,s}} = ||\{2^{pj}\phi_j * u\}||_{l^s_j L^r_x} = (\sum_j (\int 2^{pjr}|\phi_j * u|^r dx)^{\frac{s}{r}})^{\frac{1}{s}}.$$

$$||u||_{\dot{F}^p_{r,s}} = ||\{2^{pj}\phi_j * u\}||_{L^r_x l^s_j} = (\int (\sum_j 2^{pjs}|\phi_j * u|^s)^{\frac{r}{s}} dx)^{\frac{1}{r}}$$

It's a good precaution to remark that this definition holds for $p \leq 0$ too. In fact, working in homogeneous spaces saves rescaling properties, but a new problem arises: these spaces don't inherit the natural nonhomogeneous norm since some nonzero elements (like polynomials, or better, exactly polynomials) have norm equal to 0. We shall solve this obstacle in Appendix B by defining an equivalence relation - a quotient - over Sobolev space (and over these new spaces too since they get all the good properties by embedding reasons). So, all our efforts starting from here are fully justified by the theory.

An easy yet very useful embedding result is:

Lemma 1.4.

$$l^s(L^r) \subset L^r(l^s) \quad \textit{per } r \geq s,$$

$$l^s(L^r) \supset L^r(l^s) \quad \textit{per } r \leq s.$$

Proof. Let's prove the first row of the lemma, since the second could be proved in an analogous way.

Let f be a function (distribution) and $\{f_j\}$ the inherent associated sequence, as discussed before. We have

$$f \in l^s L^r \Rightarrow \sum_j \left(\int |f_j|^r dx\right)^{\frac{s}{r}} < +\infty.$$

For every j, in particular, we have $f_j \in L^r$, that is to say, $(f_j)^s \in L^{\frac{r}{s}}$. As $\frac{r}{s} \geq 1$ (this condition is crucial!), due to Minkowski inequality the sum of these elements is in $L^{\frac{r}{s}}$ too:

$$\sum_j |f_j|^s \in L^{\frac{r}{s}}$$

which is equivalent to

$$\int \left(\sum_j |f_j|^s\right)^{\frac{r}{s}} dx < +\infty$$

and so $f \in L^r l^s$. □

From this lemma we immediately obtain

Corollary 1.5.

$$\dot{B}^p_{r,s} \subset \dot{F}^p_{r,s} \quad \textit{per } r \geq s$$

$$\dot{B}^p_{r,s} \supset \dot{F}^p_{r,s} \quad \textit{per } r \leq s$$

The containment relation with Sobolev spaces is here declared:

Theorem 1.6 (Mikhlin-Hörmander)**.** *We set* $1 < r < +\infty$*. By writing*

$$H^p_r = \{u \in S', F^{-1}((1 + |\xi|^2)^{\frac{p}{2}} \hat{u}) \in L^r\}$$

the (inhomogeneous) Sobolev space of order p *associated to* L^r*, we have*

$$\dot{H}^p_r = \dot{F}^p_{r,2}$$

The reader looking for a proof can read it at [15].

Corollary 1.7.
$$\dot{B}^p_{r,2} \subset \dot{H}^p_r \quad \textit{per } r \geq 2$$
$$\dot{B}^p_{r,2} \supset \dot{H}^p_r \quad \textit{per } 1 < r \leq 2$$

Proof. It's a direct consequence of Mikhlin-Hörmander theorem and corollary 1.5. □

Let's recall one last embedding lemma which substantially is a consequence of the Bernstein inequality:

Lemma 1.8. *Let $1 \leq r_2 \leq r_1 \leq +\infty$, $s \geq 1$ and $p_1, p_2 \in \mathbb{R}$ such that $\frac{1}{r_2} - \frac{1}{r_1} = \frac{p_2 - p_1}{n}$. Then*
$$\dot{B}^{p_2}_{r_2,s} \subset \dot{B}^{p_1}_{r_1,s}$$
and it does exist a constant C such that $||u||_{\dot{B}^{p_1}_{r_1,s}} \leq C||u||_{\dot{B}^{p_2}_{r_2,s}}$.

Proof. We have
$$\frac{1}{r_1} + 1 = \frac{1}{r_2} + 1 - \frac{p_2 - p_1}{n} = \frac{1}{r_2} + \frac{1}{p}$$
(where $\frac{1}{\bar{p}} = \frac{p_2 - p_1}{n}$ is the Hölder exponent conjugated to $\frac{1}{p}$) and so, thanks to Young inequality,
$$||\phi_j * u||_{L^{r_1}} = ||\tilde{\phi}_j * \phi_j * u||_{L^{r_1}} \leq ||\tilde{\phi}_j||_{L^p} ||\phi_j * u||_{L^{r_2}}$$

Then, by rescaling on the L^p norm of $\tilde{\phi}_j$ (and by using the conjugated exponents, since a Fourier transform is involved)
$$||\phi_j * u||_{L^{r_1}} \leq 2^{j\frac{n}{\bar{p}}} ||\tilde{\phi}_0||_{L^p} ||\phi_j * u||_{L^{r_2}} = 2^{j(p_2 - p_1)} ||\tilde{\phi}_0||_{L^p} ||\phi_j * u||_{L^{r_2}}$$
or
$$2^{jp_1} ||\phi_j * u||_{L^{r_1}} \leq ||\tilde{\phi}_0||_{L^p} 2^{jp_2} ||\phi_j * u||_{L^{r_2}}$$
and it's sufficient to remember how we defined the Besov norm and defining $C = ||\tilde{\phi}_0||_{L^p}$ so thesis is showed. □

1.2 Fractional derivative

We should observe how we can define derivatives of fractional order by using the Fourier transform:

Definition 1.9. *We define* ***(fractional) derivative of order s*** *the expression*
$$|D|^s f(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} |\xi|^s \hat{f}(\xi) d\xi$$

It's not difficult to verify that we can write

$$|D|^s f(x) = K_s * f(x), \text{ con } K_s(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}|\xi|^s d\xi$$

Also it's fast to verify that, by remembering the change of variables formula for multiple integrals, the convolutive kernel is $-(n+s)$-homogeneous: so, given $\lambda \in \mathbb{R}$, say $\bar{\lambda} = \lambda$,

$$K_s(\lambda x) = \int_{\mathbb{R}^n} e^{i\langle \lambda x,\xi\rangle}|\xi|^s d\xi =$$

$$= \int_{\mathbb{R}^n} e^{i\langle x,\bar{\lambda}\xi\rangle}|\lambda\xi|^s \lambda^{-s} d(\lambda\xi)\lambda^{-n} = \lambda^{-(n+s)}K_s(x)$$

In general, one should always remember the decomposition

$$|D|^s = |D|^{s-1}|D|^1,$$

where, by the well-known properties of Fourier transform, $|D|^1 f(x) = -iH(x)\partial_x$ se $n = 1$ and $|D|^1 f(x) = \hat{H}(x)\nabla$ if $n > 1$ (where $\hat{H}(x) = sign(x)$).

Later in this section we will need to quote two interesting operators whose use is typical of harmonic analysis:

Definition 1.10. *Let $f \in L^1(\mathbb{R})$ and $I \in \mathbb{R}$ an open interval of lenght $|I|$; we call **Hardy-Littlewood's maximal operator** the operator M, where*

$$(Mf)(x) = \sup_{x\in I} \frac{1}{|I|}\int_I |f(y)|dy$$

It's not difficult to show that:

Proposition 1.11. *Hardy-Littlewood's maximal operator M is L^p-bounded $\forall \quad 1 < p \leq +\infty$.*

Definition 1.12. *Let $\hat{\phi}_j$ be the Littlewood-Paley diadic decomposition on $\mathbb{R} - \{0\}$ we introduced in the previous section; we call **Littlewood-Paley's maximal operator** the operator S, where*

$$(Sf)(x) = \sum_k |f * \phi_k|^2$$

About this last one, these is an important result whose proof requires some effort and knowledge of harmonic analysis tools. It does adfirm that these operators are L^p-bounded in such a sense:

Theorem 1.13 (Littlewood-Paley). *$\forall \quad 1 < p < \infty$ it does exist a constant $C = C(p)$ such that $\forall\ f \in S$*

$$\frac{||f||_p}{C} \leq ||Sf||_p \leq C||f||_p$$

We leave the proof of both these facts to [19].

Let's broach with a technical lemma which will be useful soon:

Lemma 1.14. *Let $f \in S$ be a Schwartz function such that $\hat{f}$ has compact support with $0 \in supp(f) := I$. Then $\forall\ \alpha > 0$ this decay estimate holds:*

$$||D|^\alpha f(x)| \leq \frac{C}{(1+|x|)^{\alpha+1}}$$

Proof. Let's prove this fact in dimension $n = 1$ for the sake of ease. By simply writing the definition of fractional derivative, we find that $||D|^\alpha f(0)|$ is bounded, since $\alpha > 0$ and $\hat{f}$ has compact support. We must bound $||D|^\alpha f(x)|$ far from 0.

$$|D|^\alpha f(x) = \int_{\mathbb{R}} e^{ix\xi}|\xi|^\alpha \hat{f}(\xi)d\xi = \frac{1}{(ix)^{[\alpha]+1}} \int_I \frac{\partial^{[\alpha]+1}}{\partial\xi^{[\alpha]+1}} e^{ix\xi}|\xi|^\alpha \beta(\xi)d\xi$$

where $[\alpha]$ denotes the floor of α and β is a C^∞ function with compact support and 0 on the boundary of I (which is only a simpler way to remember of $\hat{f}$).

Now let's rewrite

$$\frac{1}{(ix)^{[\alpha]+1}} \int_I \frac{\partial^{[\alpha]+1}}{\partial\xi^{[\alpha]+1}} e^{ix\xi}|\xi|^\alpha \beta(\xi)d\xi = \frac{1}{(ix)^{[\alpha]+1}} \int_I u^{([\alpha]+1)}v$$

where $u = e^{ix\xi}$ and $v = |\xi|^\alpha \beta(\xi)$ and we integrate by parts $[\alpha]+1$ times: the intermediate pieces of this expression are all 0 thanks to the property of nullity on the boundary of β, then

$$\frac{1}{(ix)^{[\alpha]+1}} \int_I \frac{\partial^{[\alpha]+1}}{\partial\xi^{[\alpha]+1}} e^{ix\xi}|\xi|^\alpha \beta(\xi)d\xi = \frac{(-1)^{[\alpha]+1}}{(ix)^{[\alpha]+1}} \int_I e^{ix\xi} \frac{\partial^{[\alpha]+1}}{\partial\xi^{[\alpha]+1}}(|\xi|^\alpha \beta(\xi))d\xi =$$

and $\frac{\partial^{[\alpha]+1}}{\partial\xi^{[\alpha]+1}}(|\xi|^\alpha \beta(\xi))$ is integrable over I because the singularity in 0 of $|\xi|$ is under control (this shall be true no more if we would have integrated $[\alpha]+2$ times).

Finally, by recalling the first part, using absolute values and by Riesz-Thorin interpolation, we complete the proof. □

Now we are ready to prove (always in dimension $n = 1$ for the sake of ease) a fractional chain rule (or fractional Leibniz rule):

Theorem 1.15 (Kato-Ponce)**.** *Given f, g regular enough, $\forall\ \alpha > 0$, $p_i, q_i > 1$ with $\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{r}$ and $\frac{1}{1+\alpha} < r < +\infty$ we have*

$$|||D|^\alpha(fg)||_r \leq C_f|||D|^\alpha(f)||_{p_1}||g||_{q_1} + C_g|||D|^\alpha(g)||_{p_2}||f||_{q_2}$$

that is to say, the extreme members of the derivatives bound the derivative of the entire product.

Proof. Before beginning, let's point out at the fact that the case $r > 1$ will be more than sufficient since we are going to work always and only in enough-regular spaces, say, not less than Banach (and L^r are no more Banach when $r < 1$).

Let's rewrite f e g in an opportune way by using the Paley-Littlewood decomposition:

$$\hat{f} = \hat{f} \cdot 1 = \hat{f} \sum_k \hat{\phi}_k = \sum_k \hat{f}\hat{\phi}_k \Rightarrow f = \sum_k (f * \phi_k)$$

and g admits an analogous decomposition. All this to write

$$fg = \sum_{k_1} \sum_{k_2} (f * \phi_{k_1})(g * \phi_{k_2}) =$$

$$= \underbrace{\sum_{k_1 << k_2} (f * \phi_{k_1})(g * \phi_{k_2})}_{\Sigma(k_1 << k_2)} + \underbrace{\sum_{k_1 >> k_2} (f * \phi_{k_1})(g * \phi_{k_2})}_{\Sigma(k_1 >> k_2)} + \underbrace{\sum_{k_1 \approx k_2} (f * \phi_{k_1})(g * \phi_{k_2})}_{\Sigma(k_1 \approx k_2)}$$

where $k_1 << k_2$ means that there exist a $M > 0$ *big* enough such that $k_1 \leq k_2 - M$.

Now, the first two pieces $\Sigma(k_1 << k_2)$ and $\Sigma(k_1 >> k_2)$ can be treated in the same way and in an about easily one; we can proceed with the first one, for example, by writing it:

$$\Sigma(k_1 << k_2) = \sum_{k_2} (\sum_{k_1 << k_2} f * \phi_{k_1})(g * \phi_{k_2}) =$$

$$= \sum_{k_2} ((f * \gamma_{k_2})(g * \phi_{k_2})) = \sum_{k_2} [(f * \gamma_{k_2})(g * \phi_{k_2})] * \tilde{\phi}_{k_2} = P(f, g),$$

where γ_k and $\tilde{\phi}_k$ are C^∞ functions such that

$$supp(\hat{\gamma}_k) \subset [-2^{k-M}, 2^{k-M}] \ , \ supp(\hat{\tilde{\phi}}_k) \subset [-2^{k+2}, -2^{k-2}] \cup [2^{k-2}, 2^{k+2}]$$

Now, we will show that paraproducts absorb derivatives in their own arguments. Let's introduce, for the sake of ease, two new functions η, $\tilde{\eta}$ such that

$$\widehat{\tilde{\eta}(\xi)_k} = \hat{\tilde{\phi}}_k(\xi)|\xi|^\alpha 2^{-\alpha k}$$
$$\widehat{\eta}(\xi)_k = \hat{\phi}_k(\xi)|\xi|^{-\alpha} 2^{\alpha k}$$

The crucial fact that allows us to do all the work - and this strongly marks this situation from $\Sigma(k_1 \approx k_2)$ - is that $0 \notin supp(\hat{\phi})$ and so $\hat{\eta}$ is a smooth function which allows derivative exchanges among convolutions.

Now, we get

$$|D|^\alpha P(f, g) = |D|^\alpha \sum_k [(f * \gamma_k)(g * \phi_k)] * \tilde{\phi}_k = \text{ derivative of a convolution}$$

$$= \sum_k [(f * \gamma_k)(g * \phi_k)] * |D|^\alpha \tilde{\phi}_k = \text{ definition of } \tilde{\eta}$$

$$= \sum_k [(f * \gamma_k)(g * \phi_k)] * 2^{\alpha k} \tilde{\eta}_k =$$

$$= \sum_k [(f * \gamma_k)(g * 2^{\alpha k} \phi_k)] * \tilde{\eta}_k = \text{ definition of } \eta$$

$$= \sum_k [(f * \gamma_k)(g * |D|^\alpha \eta_k)] * \tilde{\eta}_k = \text{ derivative of a convolution}$$

$$= \sum_k [(f * \gamma_k)(|D|^\alpha g * \eta_k)] * \tilde{\eta}_k = \tilde{P}(f, |D|^\alpha g)$$

and so we unloaded the weight of the derivative of the product on any of the two factors, this is a clue that we are going through the right way.

The matter about $\Sigma(k_1 \approx k_2)$ is more delicate and this same construction can not work anymore like before, due to a support issue: by the way, we can bypass this obstacle by using a decay estimate directly following by lemma 1.14

$$|\sum_k \tilde{\eta}_k * \phi_k(x)| \leq C(1 + |x|)^{-1-\alpha}$$

but this requires much more struggle (see [20]), anyway we can treat $\Sigma(k_1 \approx k_2)$ like the other two pieces and unload the weight of the derivatives, this time too.

Finally we can conclude. Let p, q with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$:

$$||P(f,g)||_r = \left| \int_{\mathbb{R}} P(f,g)(x)h(x)dx \right| =$$

$$= \left| \int_{\mathbb{R}} \sum_k [(f * \gamma_k)(x)(g * \phi_k)(x)(h * \tilde{\phi}_k)(x)] \right| \leq \text{ absolute values get inside integrals}$$

$$\leq \int_{\mathbb{R}} \sum_k |f * \gamma_k(x)||g * \phi_k(x)||h * \tilde{\phi}_k(x)| \leq \text{ Cauchy-Schwarz inequality}$$

$$\leq \int_{\mathbb{R}} (\sup_k |f * \gamma_k(x)|) \sqrt{\sum_k |g * \phi_k(x)|^2} \sqrt{\sum_k |h * \tilde{\phi}_k(x)|^2} \leq$$

$$\leq C \int_{\mathbb{R}} M[f(x)]S[g(x)]S[h(x)]dx,$$

where h is a function with $||h||_{r'} = 1$ from wich the measure on $\mathbb{R}$ depends, M the Hardy-Littlewood's max. operator and S Littlewood-Paley's max. operator (the square root, to be precise). Since all these operators are bounded on L^s, $\forall\ s > 1$, Hölder inequality let us conclude:

$$||P(f,g)||_r \leq \tilde{C}||f||_p||g||_q.$$

□

In truth, this theorem is a very particular case of a larger and complex one, the notorious Coifman-Meyer theorem. We can generalize the previous construction as following:

Definition 1.16. *Let J a set (of finite cardinality) made of diadic intervals $I_k = [2^k, 2^{k+1}]$ (and then whose lenght is $|I_k| = 2^k$). The bilinear expression*

$$P_J(f,g) = \sum_{I \in J} \frac{C_I}{\sqrt{|I|}} \langle f, \gamma_I \rangle \langle g, \phi_I \rangle \langle h, \tilde{\phi}_I \rangle$$

where $(C_I)_{I \in J}$ is a bounded sequence of elements in $\mathbb{C}$ and the functions γ_I, ϕ_I, $\tilde{\phi}_I$ have supports like in the previous theorem, is called ***discrete linear paraproduct****.*

The important result, whose very long and difficult proof can be read in [20], is the following:

Theorem 1.17 (Coifman-Meyer)**.** *Let $p, q \geq 1$. Each discrete linear paraproduct defines a mapping $(f,g) \to P_J(f,g)$ of $L^p \times L^q$ in L^r if and only if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $r \geq 1$.*

1.3 The TT^* method

As we have seen until now, all what was presented and all what will be presented has the aim to make clearer the situation about the Cauchy problem

$$(W) \begin{cases} (\partial_t^2 - \Delta_{\mathbb{R}^n})u = f \\ u(0,x) = u_0(x) \\ \partial_t u(0,x) = u_1(x) \end{cases}$$

We are now going to define the operators

$$\omega = \sqrt{-\Delta},$$

$$U(t) = e^{i\omega t},$$

$$K(t) = \omega^{-1} \sin(\omega t)$$

(so that $\dot{K}(t) = \cos(\omega t)$ and $\ddot{K}(t) = -\omega \sin(\omega t) = -\omega^2 K(t) = K(t)\Delta$).

If we analyze the question from a functional point of view, a solution of (W) is given by $u = v + w$, which are respectively solutions of the homogeneous problem with the same initial data (O) and of the inhomogeneous one with null initial data (I):

$$(H) \begin{cases} (\partial_t^2 - \Delta_{\mathbb{R}^n})v = 0 \\ v(0,x) = u_0(x) \\ \partial_t v(0,x) = u_1(x) \end{cases} \qquad (I) \begin{cases} (\partial_t^2 - \Delta_{\mathbb{R}^n})w = f \\ w(0,x) = 0 \\ \partial_t w(0,x) = 0 \end{cases} .$$

Moreover, we are able to give an explicit expression of v e w: in fact, it's easy to verify that

$$v(t,x) = \dot{K}(t)u_0(x) + K(t)u_1(x)$$

$$\partial_t v(t,x) = K(t)\Delta u_0(x) + \dot{K}(t)u_1(x)$$

is solution of (H), while, focusing only the time variable,

$$w(t,x) = \int_0^t K(t-s)f(s)ds = (K_R *_t \chi_+ f)(t)$$

$$\partial_t w(t,x) = (\dot{K}_R *_t \chi_+ f)(t)$$

is solution (for positive times) of (I), where we denoted

$$K_R(t) = \chi_+(t)K(t) = \begin{cases} K(t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}.$$

Such an operator is said to be **retarded**. In an analogous way,

$$K_A(t) = \chi_-(t)K(t) = \begin{cases} K(t) & \text{if } t \leq 0 \\ 0 & \text{if } t > 0 \end{cases}$$

is said to be **anticipated**.

The following theorems should answer some questions about a functional interpretation of the wave equation. Let's recall the TT^* method as presented in [13]. We will consider a Banach space X and its dual X^* (eventually non trivial). Let $D \subset X$ vectorial subspace which is dense in X (so that we have $X^* \subset D^*$). Finally, let H be a Hilbert space (tipically $H = L^2(\Omega)$) and $T : D \to H$ a linear application: so the adjoint application $T^* : H \to D^*$ is defined by

$$\langle T^*h, f\rangle_D = \langle h, Tf\rangle \ \forall\ f \in D, h \in H.$$

Lemma 1.18. *These three facts are equivalent:*

1. *It does exist a real number $a > 0$ such that $\forall\ f \in D$ it holds*

$$||Tf||_H \leq a||f||_X;$$

2. *$R(T^*) \subset X^*$ and it does exist a real number $a > 0$ such that $\forall\ h \in H$ it holds*

$$||T^*h||_{X^*} \leq a||h||_H;$$

3. *$R(T^*T) \subset X^*$ and it does exist a real number $a > 0$ such that $\forall\ f \in D$ it holds*

$$||T^*Tf||_{X^*} \leq a^2||f||_X.$$

The number a is the same for all the sentences. If one of the three is satisfied, the operators T e T^*T can be prolonged by continuity to bounded linear operators from X in H and X^* respectively.

Proof. 1) $\Rightarrow$ 2) We take $h \in H$. Then, we have $\forall\ f \in D$,

$$|\langle T^*h, f\rangle_D| = |\langle h, Tf\rangle_H| \leq ||h||_H \cdot ||Tf||_H \leq a \cdot ||h||_H \cdot ||f||_X$$

and now it's sufficient to choose f with $||f||_X = 1$. We notice that the condition $R(T^*) \subset X^*$ means that we can compute the norm $||T^*h||_{X^*}$.

2) $\Rightarrow$ 1) We take $f \in D$. Then, we have $\forall\ h \in H$,

$$|\langle h, Tf\rangle_H| = |\langle T^*h, f\rangle_D| \leq ||T^*h||_{X^*} \cdot ||f||_X \leq a \cdot ||h||_H \cdot ||f||_X$$

and now it's sufficient to choose h with $||h||_H = 1$.

1) $\Rightarrow$ 3) We take $f \in D$. Then, we have $\forall\ g \in D$,

$$|\langle T^*Tf, g\rangle_D| = |\langle Tf, Tg\rangle_H| \leq ||Tf||_H \cdot ||Tg||_H \leq a^2 \cdot ||f||_X \cdot ||g||_X$$

and now it's sufficient to take g with $||g||_X = 1$.

3) $\Rightarrow$ 1) we take $f \in D$. Then

$$||Tf||_H^2 = \langle Tf, Tf\rangle_H = \langle T^*Tf, f\rangle_D \leq ||T^*Tf||_{X^*} \cdot ||f||_X \leq a^2 \cdot ||f||_X^2$$

□

This lemma can be used to mix different applications among different spaces, in the following sense:

Corollary 1.19. *We consider a Hilbert space H, a vectorial space D dense in X_1, X_2 and two triplets (X_1, T_1, a_1) and (X_2, T_2, a_2) satisfying the conditions of the previous lemma. Then, for every choice of indices $\{i, j\} \in \{1, 2\}$ vale $R(T_i^*T_j) \subset X_i^*$ e $\forall\ f \in D$*

$$||T_i^*T_jf||_{X_i^*} \leq a_i a_j ||f||_{X_j}$$

Proof. This proof is very simple. we take $f \in D$. Then, we have $\forall\ g \in D$,

$$|\langle T_i^*T_jf, g\rangle_D| = |\langle T_jf, T_ig\rangle_H| \leq ||T_jf||_H \cdot ||T_ig||_H \leq a_j||f||_{X_j} \cdot a_i||g||_{X_i}$$

where the last inequality follows from the fact that the triplets (X_1, T_1, a_1) and (X_2, T_2, a_2) satisfy the conditions of the previous lemma (like the first one, in particular). The, it's sufficient to take g with $||g||_{X_i} = 1$. □

We can show a tangible application of this method. With $L^p(I, L^q)$ we mean the space of measurable functions from f to I in L^q such that $||f||_{L^q} \in L^p(I)$. Let's consider a Hilbert space H, a one-parameter unitary group U on H, I a interval over $\mathbb{R}$. We define the bounded operator $A : L^1(I, H) \to H$

$$A(f) = \int_I U(-s)f(s)ds = (U *_t f)(0).$$

This operator is bounded as U is a unitary group:

$$||Af||_H \leq a||f||_{L^1} \text{ con } a \leq 1$$

and so the conditions of the TT^* lemma are verified with $X = L^1(I, H)$, $a = 1$ and $D \subset X$ its (whatever) dense subspace.

The adjoint operator $A^* : H \to L^\infty(I, H)$ should be such that

$$\langle Af, h\rangle = \int_I U(-s)f(s)\bar{h}ds = \int_I f(s)\overline{U(s)h}ds = \langle f, A^*h\rangle$$

where the first equality is obtained by apllying the inner product on H, whereas the last one is the duality product on $L^2(I,H)$. In particular, $(A^*h)(t) = U(t)h$.

We can consequently define the composed operator (which is bounded too), $A^*A : L^1(I,H) \to L^\infty(I,H)$ defined by

$$(A^*Af)(t) = A^*(\int_I U(-s)f(s)ds)(t) = U(t)\int_I U(-s)f(s)ds =$$

$$= \int_I U(t-s)f(s)ds = (U *_t f)(t)$$

If one observes the solution v of the problem (H), he can notice a similarity with the operator A^* of this example; meanwhile, the solution w of (I), makes a similarity with the composition A^*A. To be really coherent with what has written until now, however, we need the operator

$$(A^*A)_R f(t) = (U_R *_t f)(t)$$

but, to this point, nobody can guarantee the pertinence of the TT^* lemma since we compromised the writing of the operator as a composition of one application with its adjoint: one should effort to understant if, in some way, the previous estimates can be recovered in this case.

In some of these cases, nevertheless, a portion of this problem can be bypassed by some interpolation considerations. The right spaces to apply these matters are of this kind:

Definition 1.20. *A function (distribution) space X of the variables (t,x) is said* ***stable under time-restriction*** *if the product with the characteristic function of a time interval J is a bounded operator in X uniformly in respect to J.*

In our case, we will always consider spaces like $X = L^p(I,Y)$ with Y distribution space of the variable$x$, so this condition is quietly respected.

Lemma 1.21. *Let H be a Hilbert space, I a time interval on $\mathbb{R}$, $X \subset S'(I \times \mathbb{R}^n)$ a Banach space which is stable under time-restriction and A a convolutive operator (like the one defined in the previous example) where its one-parameter group associated is unitary. We also suppose that X and A satisft any of the condition of the TT^* lemma. Then the operator $(AA^*)_R$ can be extent to a bounded operator from $L^1(I,H)$ to X^* (and, by duality, from X to $L^\infty(I,H)$).*

Proof. Let $f \in X$. Then

$$||(A^*A)_R f(t)||_H = ||A\chi_+(t-\cdot)f||_H \leq a\cdot \sup_{t\in I} ||\chi_+(t-\cdot)||_{B(X)}||f||_X$$

where the equality is due to the writing of $(A^*A)_R$ and to the associated group being unitary, whereas the upper bound follows from TT^* lemma and stability under time-restriction. □

Corollary 1.22. *Let* $X_\theta, 0 \leq \theta \leq 1$ *a collection of Banach spaces, with* $X_0 = L^1(I, H)$ *and* $X_1 = X$ *such that* (X_θ, X_θ^*) *be an interpolation between* $(X_0, X_0{}^*)$ *and* $(X_1, X_1{}^*)$. *Let's suppose that* A, X *satisfy the hypothesis of the previous lemma and* $(A^*A)_R$ *is bounded from* X *to* X^*. *Then* $(A^*A)_R$ *is bounded from* $X_{\bar\theta}$ *to* X_θ^* $\forall\ 0 \leq \bar\theta, \theta \leq 1$.

Proof. $(A^*A)_R$ is trivially bounded from X_0 to X_0^*, from X_1 to X_1^* by hypothesis, and from X_0 to X_1^* (and from X_1 to X_0^*) by using the previous lemma. An interpolation argument concludes the proof. □

Chapter 2

Local existence and uniqueness

We now prove the existence of a time interval $[0,T]$ of existence and unicity of a weak solution (which is yet to define in a proper way) of the equation

$$(*)\begin{cases} (-\partial_t^2+\Delta_{\mathbb{R}^n})u = F(u,u_t,Du) \\ u(0,x)=g(x) \\ \partial_t u(0,x) = h(x) \end{cases}$$

The most interesting case we'll consider it's $F(u,u_t,Du) = u|u|^{p-1}$ for some positive integer p; anyway, a crucial fact is that $F(0,0,0) = 0$ (that's true, in particular, for the case we'll consider).

We'll need some sophisticated Sobolev inequalities, which will make the proof of the theorem easier:

Lemma 2.1. *Supposing we have $u_1, ..., u_m \in H^k(\mathbb{R}^n)$, con $k > \frac{n}{2}$:*

1. *Given indices $\beta_1, ..., \beta_m \in \mathbb{N}$, $\sum \beta_i \leq k$, we have*

$$||\prod_{i=1}^m D^{\beta_i}u_i||_{L^2(\mathbb{R}^n)} \leq C(k,m,n)\prod_{i=1}^m ||u_i||_{H^k(\mathbb{R}^n)};$$

2. *(Moser) Let $f \in C^\infty(\mathbb{R}^m)$ be a function with the condition $f(0) = 0$. Then $f \circ U = f(u_1, ..., u_m) \in H^k(\mathbb{R}^n)$ and moreover*

$$||f||_{H^k(\mathbb{R}^n)} \leq \phi(||u_1||_{H^k(\mathbb{R}^n)}, ..., ||u_m||_{H^k(\mathbb{R}^n)}),$$

where ϕ is some continuous nondecreasing (in every argument) function depending by f,k,m,n.

Proof. To prove the first fact, we must remember the following Sobolev inequalities (Appendix B; anyway, a proof can be read in [9]):

$$||D^\beta u||_{L^{p_i}} \leq C_i(n,k)||u||_{H^k} \text{ where } (@)\begin{cases} p_i = \infty \text{ if } \frac{1}{2}+\frac{|\beta_i|}{n} < \frac{k}{n} \\ 2 \leq p_i < \infty \text{ if } \frac{1}{2}+\frac{|\beta_i|}{n} = \frac{k}{n} \\ \frac{1}{p_i} = \frac{1}{2} - \frac{k-|\beta_i|}{n} \text{ if } \frac{1}{2}+\frac{|\beta_i|}{n} > \frac{k}{n} \end{cases}$$

We suppose now to dispose the β_i (with a proper multiindex order, for example GREVLEX) in a way that

$$\beta_1 \geq \beta_2 \geq ... \geq \beta_\alpha \geq ... \geq \beta_\gamma \geq ... \geq \beta_m$$

where the first multiindices α are involved in the third estimate of (@), the ones from $\alpha + 1$ to γ in the second estimate, all the remaining ones in the first estimate.

It's easy to notice that we are free to choose m numbers p_i so that $\sum_{i=1}^{m} \frac{1}{p_i} = \frac{1}{2}$: in fact the conditions $k > \frac{n}{2}$ e $\sum \beta_i \leq k$ guarantee that $p_1, ..., p_\alpha$ are not excessively small. More precisely,

$$\frac{1}{p_1} + ... + \frac{1}{p_\alpha} = \frac{\alpha}{2} - \frac{\alpha k}{n} + \frac{\sum_{i=1}^{\alpha} |\beta_i|}{n} \leq \frac{\alpha}{2} - \frac{(\alpha - 1)k}{n} < \frac{\alpha}{2} - \frac{(\alpha - 1)}{2} < \frac{1}{2}$$

and now it's clear that the remaining indices $p_{\alpha+1}, ..., p_m$ can be freely chosen small enough in order to make the sum correct.

This is important as it allows us to apply the Hölder inequality:

$$||f||_{L^p} \leq ||f_1||_{L^{p_1}} ... ||f_m||_{L^{p_m}}$$

where we have $f = f_1 ... f_m$ and each $f_i \in L^{p_i}$. In this specific case, $p = 2$ and so

$$|| \prod_{i=1}^{m} D^{\beta_i} u_i ||_{L^2(\mathbb{R}^n)} \leq \prod_{i=1}^{m} ||D^{\beta_i} u_i||_{L^{p_i}(\mathbb{R}^n)} \leq$$

$$\leq \prod_{i=1}^{m} C_i(n, k) ||u||_{H^k(\mathbb{R}^n)} = C(k, m, n) \prod_{i=1}^{m} ||u_i||_{H^k(\mathbb{R}^n)}$$

as wished.

Coming to Moser theorem now; we must remember that when $k > \frac{n}{2}$ the following Sobolev inequality holds:

$$||u||_{L^\infty(\mathbb{R}^n)} \leq C ||u||_{H^k(\mathbb{R}^n)}$$

We must estimate the L^2-norms of the multiindex derivatives of f: the generic term $D^\alpha f(u_1, ..., u_m)$ can be written as a sum of terms like

$$A \cdot D^{\beta_1} u_1 ... D^{\beta_m} u_m,$$

where $0 < |\beta_i| \leq |\alpha|$ e $\sum_i \beta_i = \alpha$ and A depends by f and by its partial derivatives evaluated in u_i. By applying the last Sobolev inequality, $||A||_{L^\infty(\mathbb{R}^n)}$ is bounded.

Now,

$$||A \cdot D^{\beta_1} u_1 ... D^{\beta_m} u_m||_{L^2(\mathbb{R}^n)} \leq ||A||_{L^\infty(\mathbb{R}^n)} \cdot ||D^{\beta_1} u_1 ... D^{\beta_m} u_m||_{L^2(\mathbb{R}^n)}$$

and thanks to the first assertion of the lemma the righthand member of this equality is bounded by an expression involving only the Sobolev norms of u_i.

Finally, we can repeat the same consideration to each term $D^\alpha f$ and summing all over (the case $\alpha = 0$ need the hypothesis $f(0) = 0$). $\square$

Now, as we promised before, let's talk a bit about well-posedness of the problem and weak solution:

Definition 2.2. *We hereby declare that the problem* $\partial_t \boldsymbol{u} - A\boldsymbol{u} = F(\boldsymbol{u})$ *il* ***wellposed*** *(by the mean of Hadamard definition) in* $H^k(\mathbb{R}^n)$ *if* $\forall\ R > 0 \exists\ T(R) > 0$ *such that* $\forall\ \boldsymbol{u}(0,x) = f \in B(R) := \{y \in H^k(\mathbb{R}^n), ||y||_{H^k(\mathbb{R}^n)} \leq R\}$ *it does exist a unique solution* $\boldsymbol{u}(t,x) \in C([0,T], H^k(\mathbb{R}^n))$ *such that the flux map*

$$f \rightarrow \begin{cases} \partial_t \boldsymbol{u} - A\boldsymbol{u} = F(\boldsymbol{u}) \\ \boldsymbol{u}(0,x) = f(x) \end{cases}$$

(which takes an initial condition $f \in B(R)$ *and brings it in the inherent solution in* $C([0,T], H^k(\mathbb{R}^n))$*).* $\boldsymbol{u}(t,x)$ *is continuous. The image of such a map is said* ***weak solution*** *of the problem when it's written in its integral form*

$$\boldsymbol{u}(t,x) = e^{At}\boldsymbol{u}(0,x) + \int_0^t e^{A(t-s)} F(\boldsymbol{u}) ds$$

This definition here admits a vectorial reading, so the dimension of the problem and the amount of initial condition must not mislead the reader: this is a perfectly pertinent notion. In effect, the problem (*) admits a rewriting of this kind:

$$\partial_t \begin{pmatrix} u \\ u_t \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 \\ F(u, u_t, Du) \end{pmatrix}$$

and the choice of suitable initial conditions is made in the space $K = H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)$ with the norm

$$|| \begin{pmatrix} u \\ u_t \end{pmatrix} ||_K = ||u||_{H^k(\mathbb{R}^n)} + ||u_t||_{H^{k-1}(\mathbb{R}^n)}.$$

Further information on this kind of approach can be extracted in [16].

Theorem 2.3 (Local existence)**.** *Let* $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ *be a* C^∞*-regular function with* $F(0,0,0) = 0$*. Moreover, let* $g \in H^k(\mathbb{R}^n)$*,* $h \in H^{k-1}(\mathbb{R}^n)$ *for some* $k > \frac{n}{2} + 1$*.*

Then we can find a time $T = T(||g||_{H^k(\mathbb{R}^n)}, ||h||_{H^{k-1}(\mathbb{R}^n)})$ *so that the problem*

$$(*) \begin{cases} (-\partial_t^2 + \Delta_{\mathbb{R}^n})u = F(u, u_t, Du) \\ u(0,x) = g(x) \\ \partial_t u(0,x) = h(x) \end{cases}$$

admits one (and only one) (weak) solution u *so that* $u \in C([0,T], H^k(\mathbb{R}^n))$ *and* $u' \in C([0,T], H^{k-1}(\mathbb{R}^n))$*.*

Proof. Before starting the proof, let's precise that Moser theorem, as expressed before, let us talk about the local existence of the solution in a more general form: in the proof we're gonna provide, one could substitute the hypothesis over nonlinearity with a weaker one, like

$$F = F(u), F \in C^1 \text{ near } 0, |F(u)| + |u||F'(u)| \leq K|u|^p$$

(in effect, this is more than sufficient for the problem we're going to discuss).

The proof is based in defining a proper norm (which naturally arises from energy estimates) with whom apply an argument of contraction: from that, we'll obtain existence and unicity of a local solution.

Let's define

$$X = \{u \in C([0,T], H^1(\mathbb{R}^n)) | u' \in C([0,T], L^2(\mathbb{R}^n))\}$$

equipped with the norm

$$||u||_{1,X} = \sup_{t\in[0,T]} (||u||_{H^1(\mathbb{R}^n)} + ||u'||_{L^2(\mathbb{R}^n)})$$

and then define a stronger norm:

$$||u||_{k,X} = \sup_{t\in[0,T]} (||u||_{H^k(\mathbb{R}^n)} + ||u'||_{H^{k-1}(\mathbb{R}^n)})$$

(elements of X are bounded in the first norm, not necessarily in the second one).

So, we can define a particular subset of X constructed with the elements which are bounded in the second norm too and maybe some more request over their behaviour at $t = 0$:

$$X_\lambda = \{u \in X : ||u||_{k,X} \leq \lambda, u(0,x) = g(x), u'(0,x) = h(x)\}$$

Among the elements of X_λ we're going to look for a possible solution of $(*)$ and over this very element we're going to apply the contraction theorem.

More precisely, let's define over this space an operator A so that, taken $v \in X_\lambda$, we have $A[v] = u$ where u does solve

$$(**) \begin{cases} (-\partial_t^2 + \Delta_{\mathbb{R}^n})u = F(v, v_t, Dv) \\ u(0,x) = g(x) \\ \partial_t u(0,x) = h(x) \end{cases}$$

Let's define the convenient following quantities:

$$E_k(t) = ||u(t)||^2_{H^k(\mathbb{R}^n)} + ||u'(t)||^2_{H^{k-1}(\mathbb{R}^n)}$$

$$G_k(t) = ||v(t)||^2_{H^k(\mathbb{R}^n)} + ||v'(t)||^2_{H^{k-1}(\mathbb{R}^n)}$$

Let's prove now the following energy estimates:

$$E_k(t) \leq E_k(0) + C\int_0^t \phi(G_k(\tau))d\tau,$$

where $0 \leq t \leq T$ e ϕ is some continuous nonincreasing function depending of n, k, F.

Chosen a random $\alpha, |\alpha| < k$, calling $w = D^\alpha u$ we apply D^α a $(**)$:

$$-w_{tt} + \Delta w = D^\alpha(F(v, v_t, Dv))$$

and then obtaining

$$\frac{d}{dt}\int_{\mathbb{R}^n}(w_t^2 + |Dw|^2)dx = \int_{\mathbb{R}^n}(2w_t w_{tt} + 2Dw \cdot Dw_t)dx =$$

(using that $\langle Dw, Dw_t\rangle = \langle -\Delta w, w_t\rangle$)

$$= 2\int_{\mathbb{R}^n}(w_{tt} - \Delta w)w_t dx = -2\int_{\mathbb{R}^n} D^\alpha(F(v, v_t, Dv))w_t dx \leq$$

(using that $-2ab \leq a^2 + b^2$)

$$\leq \int_{\mathbb{R}^n}(w_t^2 + |D^\alpha(F(v, v_t, Dv))|^2)dx \leq$$

(using Moser theorem)

$$\leq \int_{\mathbb{R}^n}(w_t^2 + C\tilde{\phi}^2(||v||_{H^{k-1}(\mathbb{R}^n)}, ||v_t||_{H^{k-1}(\mathbb{R}^n)}, ||v_1||_{H^{k-1}(\mathbb{R}^n)}, ..., ||v_n||_{H^{k-1}(\mathbb{R}^n)})dx \leq$$

$$\leq \int_{\mathbb{R}^n} w_t^2 dx + \phi(G_k(t))$$

and finally, we can conclude by applying Gronwall's lemma, integrate over time and summing over all the multiindices $|\alpha| < k$ to obtain the energy estimate.

Let's show now that, chosen a propere bound λ and a maximal time T, the operator A is a contraption. Before all, let's see that

$$A : X_\lambda \to X_\lambda.$$

The energy estimate - in its integral form - we just obtained allows us to write

$$||u||_{k,X}^2 \leq ||g||_{H^k(\mathbb{R}^n)}^2 + ||h||_{H^{k-1}(\mathbb{R}^n)}^2 + CT\phi(||v||_{k,X}^2)$$

where ϕ is the nonincreasing function provided by Moser theorem.

The work is to choose λ big enough as well as T small enough: for example,

$$\lambda^2 = 2(||g||_{H^k(\mathbb{R}^n)}^2 + ||h||_{H^{k-1}(\mathbb{R}^n)}^2)$$

and T is such that

$$2CT\phi(||v||_{k,X}^2) \leq \lambda^2.$$

At this point, we stated that

$$||A(v)||_{k,X}^2 = ||u||_{k,X}^2 \leq \frac{\lambda^2}{2} + \frac{\lambda^2}{2} \leq \lambda^2$$

and so $A : X_\lambda \to X_\lambda$.

Now let's show that this is a contraption by definition, that is to say, chosen $v_a, v_b \in X_\lambda$

$$||A(v_a) - A(v_b)||_{1,X} \leq \frac{1}{2}||v_a - v_b||_{1,X}$$

We can write $u_a = A(v_a)$ e $u_b = A(v_b)$ e $u = u_a - u_b$. By repeating - more or less at the same way - the previous computations,

$$\frac{d}{dt}\int_{\mathbb{R}^n}(u^2 + u_t^2 + |Du|^2)dx =$$

$$= -2\int_{\mathbb{R}^n}(F(v_a, (v_a)_t, Dv_a) - F(v_b, (v_b)_t, Dv_b) - u)u_t dx \leq$$

$$\leq \int_{\mathbb{R}^n}(u_t^2 + u^2)dx + C\int_{\mathbb{R}^n}(|v_b - v_a|^2 + |(v_b)_t - (v_a)_t|^2 + |Dv_b - Dv_a|^2)dx$$

where C is a costant depending of v_a, v_b and all its derivatives and bounded in the infinite norm.

Once again, by the mean of the Gronwall's lemma, integrating over time, we manage to write

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}^n}(u^2+u_t^2+|Du|^2)dx \leq C\int_0^T dt\int_{\mathbb{R}^n}(|v_b-v_a|^2+|(v_b)_t-(v_a)_t|^2+|Dv_b-Dv_a|^2)dx \leq$$

$$\leq CT||v_a - v_b||_{1,X}^2$$

that is exactly

$$||A(v_a) - A(v_b)||_{1,X} \leq CT||v_a - v_b||_{1,X}$$

and we conclude by choosing $T < \frac{1}{2C}$.

Finally, the contraption theorem guarantees existence and unicity of an element X_λ u such that $A(u) = u$, which is exactly a solution of $(*)$. $\square$

Chapter 3

Strichartz inequalities

Let's recall the expression of the solutions of the wave equation when decomposed in its homogeneous and null initial data parts:

$$v(t,x) = \dot{K}(t)u_0(x) + K(t)u_1(x)$$

$$\partial_t v(t,x) = K(t)\Delta u_0(x) + \dot{K}(t)u_1(x)$$

$$w(t,x) = \int_0^t K(t-s)f(s)ds = (K_R *_t \chi_+ f)(t)$$

$$\partial_t w(t,x) = (\dot{K}_R *_t \chi_+ f)(t)$$

Let's also suppose that the initial data u_0, u_1 of (W) are respectively $\dot{H}^s$ and $\dot{H}^{s-1}$ regular, and we concisely write

$$(u_0, u_1) \in Y^s := \dot{H}^s \oplus \dot{H}^{s-1}.$$

To be cautious, we will recall again the useful exponents we defined in Section 1.1:

$$\beta(r) = \frac{n-1}{2}\alpha(r); \quad \gamma(r) = (n-1)\alpha(r); \quad \delta(r) = n\alpha(r)$$

and remember that when $n \geq 3$ these three quantities are at the same time in alphabetic and increasing order.

The most general formulation of the inequalities is the following:

Theorem 3.1 (Strichartz)**.** *Let $p_1, p_2 \in \mathbb{R}$ and $q_1, q_2, r_1, r_2 \geq 2$ satisfying the following conditions:*

$$\frac{2}{q_i} \leq \min(1, \gamma(r_i)) \qquad \textit{for } i = 1,2; \qquad (C1)$$

$$(\gamma(r_i), \frac{2}{q_i}) \neq (1,1) \qquad \textit{for } i = 1,2; \qquad (C2)$$

$$p_1 + \delta(r_1) - \frac{1}{q_1} = s \qquad (C3)$$

$$(p_1 + \delta(r_1) - \frac{1}{q_1}) + (p_2 + \delta(r_2) - \frac{1}{q_2}) = 1 \qquad (C4)$$

1. *By writing* $\dot{B}^{p_1}_{r_1,2} = \dot{B}^{p_1}_{r_1}$, *it holds*

$$(S1)\quad ||v||_{L^{q_1}(\mathbb{R},\dot{B}^{p_1}_{r_1})} + ||\partial_t v||_{L^{q_1}(\mathbb{R},\dot{B}^{p_1-1}_{r_1})} \leq C||(u_0,u_1)||_{Y^s}$$

2. *For every interval* $I \subset \mathbb{R}$ *(maybe* $\mathbb{R}$ *itself), it holds*

$$(S2)\quad ||K * f||_{L^{q_1}(I,\dot{B}^{p_1}_{r_1})} \leq C||f||_{L^{q_2}(I,\dot{B}^{-p_2}_{r_2})}$$

3. *For every interval* $I = [0,T), T > 0$ *(maybe* $\mathbb{R}^+$ *itself), it holds*

$$(S3)\quad ||w||_{L^{q_1}(I,\dot{B}^{p_1}_{r_1})} + ||\partial_t w||_{L^{q_1}(I,\dot{B}^{p_1-1}_{r_1})} \leq C||f||_{L^{q_2}(I,\dot{B}^{-p_2}_{r_2})}$$

To be honest, one can show an equivalent version of this theorem, involving the operator $U(t)$ defined in Section 1.3. This is made possible by remembering the expressions of the solutions v and w (and then K and $\dot{K}$). Moreover, the properties of the laplacian operator ensure that $\omega^\alpha = \sqrt{-\Delta}^\alpha$ is an isomorphism between $\dot{B}^p_r$ and $\dot{B}^p_{r-\alpha}$ for every $\alpha \in \mathbb{R}$ and so we can choose $s = 0$ in $(S1)$ without losing generality.

Theorem 3.2 (Strichartz, eq. version)**.** *Let* $p_1, p_2 \in \mathbb{R}$ *and* $q_1, q_2, r_1, r_2 \geq 2$ *satisfying the following conditions:*

$$\frac{2}{q_i} \leq \min(1,\gamma(r_i)) \qquad \text{for } i = 1,2; \qquad (C1)$$

$$(\gamma(r_i),\frac{2}{q_i}) \neq (1,1) \qquad \text{for } i = 1,2; \qquad (C2)$$

$$p_1 + \delta(r_1) - \frac{1}{q_1} = 0 \qquad (C3')$$

1. *By writing* $\dot{B}^{p_1}_{r_1,2} = \dot{B}^{p_1}_{r_1}$, *it holds*

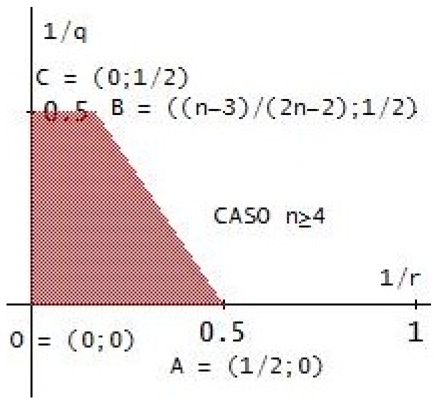
$$(S1')\quad ||U(\cdot)u||_{L^{q_1}(\mathbb{R},\dot{B}^{p_1}_{r_1})} \leq C||u||_{L^2}$$

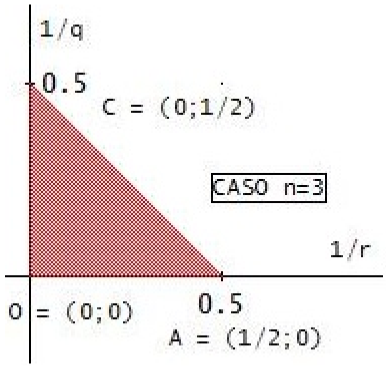
2. *For every interval* $I \subset \mathbb{R}$ *(maybe* $\mathbb{R}$ *itself), it holds*

$$(S2')\quad ||U * f||_{L^{q_1}(I,\dot{B}^{p_1}_{r_1})} \leq C||f||_{L^{\bar{q}_2}(I,\dot{B}^{-p_2}_{r_2})}$$

3. *gor every interval* $I = [0,T), T > 0$ *(maybe* $\mathbb{R}^+$ *itseld), it holds*

$$(S3')\quad ||U_R * f||_{L^{q_1}(I,\dot{B}^{p_1}_{r_1})} \leq C||f||_{L^{\bar{q}_2}(I,\dot{B}^{-p_2}_{r_2})}$$





(In the images we can see a possible representation fo the admissible couples $(\frac{1}{r}, \frac{1}{q})$ of the conditions $(C1)$ and $(C2)$ for the cases $n \geq 4$ and $n = 3$. in the first one, the point B is to be excluded; in the second, the point B (which is the same of the point C) again is to be excluded).

Before beginning, we can show how the embedding lemma 1.8 is useful in such a context. In effect, by using it, one can notice that the left member norms of $(S1)$, $(S2)$ e $(S3)$, when q_1 and s are fixed, increase when p_1 increases (or similarly $\frac{1}{r_1}$, that is to say, r_1 decreases). So one can prove the theorem for the biggest p_1 (or the smallest r_1) among the allowed choices fo the limitations.

In particular, when $q_1 > 2$, $\gamma(r_1) = \frac{2}{q_1}$ is an allowed upper bound thanks to $(C1)$ and so we can prove the teorem in this very case, in which we fix $p_1 = s - \beta(r_1)$ in the three inequalities (or similarly $p_1 = -\beta(r_1)$ in its equivalent formulation). The case $q_1 = 2$ is more delicate since $(C2)$ forbids such a choice.

For the same reason, one can do the same with the right members of $(S2)$ and $(S3)$, which are decreasing in p_2 (in $\frac{1}{r_2}$) when we fix q_2 and s fissati. when $q_2 > 2$, $\gamma(r_2) = \frac{2}{q_2}$ is allowed and again one can prove the theorem in the case $p_2 = 1 - s - \beta(r_2)$ in the two inequalities (or, similarly, $p_2 = 1 - \beta(r_2)$ in its equivalent formulation). Again, such a choice is forbidden when $q_2 = 2$.

Proof. We are going to prove only the case $q > 2$. Even if the general idea (and the proceeding way) are about the same, we leave the case $q = 2$ to [12].

We start from the stationary phase estimate

$$\sup_x \left| \int \hat{\phi}_0(\xi) e^{it|\xi| + i\langle x,\xi\rangle} d\xi \right| \leq \min(||\hat{\phi}_0||_{L^1}, C_0|t|^{-\frac{n-1}{2}})$$

(one can read [15], sec. 7.7 for a proof).

Through a rescaling $\xi \to 2^{-j}\xi$, $t \to 2^j t$, $x \to 2^j x$ and minding $\hat{\phi_j}(\xi) = \hat{\phi}_0(2^{-j}\xi)$, we have

$$\sup_x |\int \hat{\phi}_j(\xi) e^{it|\xi| + i\langle x,\xi\rangle} d\xi| \leq \min(2^{nj}||\hat{\phi}_0||_{L^1}, C_0 2^{j\frac{n+1}{2}}|t|^{-\frac{n-1}{2}})$$

and then

$$||U(t)\phi_j||_{L^\infty} \leq C \min(2^{nj}, 2^{j\frac{n+1}{2}}|t|^{-\frac{n-1}{2}}).$$

We now consider a function (distribution) $f = f(x) = f(0,x)$ (of the only space variable, or similarly space-time in $t = 0$) regular enough . Thanks to lemma 1.2, to a change of variable and Young inequality,

$$||\phi_j * U(t)f||_{L^\infty} = ||\tilde{\phi}_j * \phi_j * U(t)f||_{L^\infty} =$$

$$= ||\phi_j * U(t)\tilde{\phi}_j * f||_{L^\infty} \leq ||U(t)\phi_j||_{L^\infty}||\tilde{\phi}_j * f||_{L^1}$$

and by using what deduced few rows ago,

$$||\phi_j * U(t)f||_{L^\infty} \leq C \min(2^{nj}, 2^{j\frac{n+1}{2}}|t|^{-\frac{n-1}{2}})||\tilde{\phi}_j * f||_{L^1}.$$

By interpolating this last quantity and the L^2 estimate ($U(t)$ is unitary on it), we get for $2 \leq r \leq +\infty$

$$||\phi_j * U(t)f||_{L^r} \leq$$

$$\leq C \min(2^{nj(1/2-1/r)}, 2^{2j\frac{n+1}{2}(1/2-1/r)}|t|^{-2\frac{n-1}{2}(1/2-1/r)})||\tilde{\phi}_j * f||_{L^{\frac{1}{1-1/r}}}$$

or, more shortly,

$$||\phi_j * U(t)f||_r \leq C \min(2^{j\delta(r)}, 2^{2j\beta(r)}|t|^{-\gamma(r)})||\tilde{\phi}_j * f||_{\bar{r}}$$

We can freely choose one between the two quantities inside the minimum, since we are writing an upper bound: so we choose the second one.

$$||\phi_j * U(t)f||_r \leq C2^{2j\beta(r)}|t|^{-\gamma(r)}||\tilde{\phi}_j * f||_{\bar{r}}$$

By multiplying for $2^{-j\beta(r)}$

$$||2^{-j\beta(r)}\phi_j * U(t)f||_r \leq C|t|^{-\gamma(r)}||2^{j\beta(r)}\tilde{\phi}_j * f||_{\bar{r}}$$

Now, we take the l^2_j norm and we recall the definition of Besov norm,

$$||U(t)f||_{\dot{B}_r^{-\beta(r)}} \leq C|t|^{-\gamma(r)}||f||_{\dot{B}_{\bar{r}}^{\beta(r)}}$$

(in the right member we are using $\tilde{\phi}_j$ instead of ϕ_j, but we can do such a thing by alterating the constant C).

By adding the time dependance of f the previous expression can be read:

$$||U(t-t')f(t')||_{\dot{B}_r^{-\beta(r)}} \leq C|t-t'|^{-\gamma(r)}||f(t')||_{\dot{B}_{\bar{r}}^{\beta(r)}}$$

we consider now $\gamma(r) = \frac{2}{q} < 1$ (since $q > 2$) and $I \subset \mathbb{R}$ a time interval. By integrating in t', taking L^q time norm and applying the Hardy-Littlewood-Sobolev inequality, we get both the inequalities

$$(EQ1) \qquad ||U *_t f||_{L^q(I,\dot{B}_r^{-\beta(r)})} \leq C||f||_{L^{\bar{q}}(I,\dot{B}_{\bar{r}}^{\beta(r)})}$$

$$(EQ2) \qquad ||U_R *_t f||_{L^q(I,\dot{B}_r^{-\beta(r)})} \leq C||f||_{L^{\bar{q}}(I,\dot{B}_{\bar{r}}^{\beta(r)})}$$

the former not retarded, the latter retarded, both very similar to our aim $(S2')$ e $(S3')$ respectively: in effect, they are the diagonal cases $p_i = \beta(r_i)$, $q_1 = q_2$, $r_1 = r_2$. Now it's the time for the TT^* method.

About the first one, one can follow the example in the section 1.3 and define

$$A(f) = \int_I U(-s)f(s)ds = (U *_t f)(0).$$

$$X = L^{\bar{q}}(I, \dot{B}_{\bar{r}}^{\beta(r)})$$

By doing like that,

$$(A^*Af)(t) = \int_I U(t-s)f(s)ds = (U *_t f)(t)$$

$$X^* = L^q(I, \dot{B}_r^{-\beta(r)})$$

and so $(EQ1)$ is exactly the third condition of the TT^* lemma. So we can use the second condition with $I = \mathbb{R}$:

$$||U(\cdot)u||_{L^q(\mathbb{R}, \dot{B}_r^{-\beta(r)})} \leq C||u||_2$$

and by posing as in the extreme cases $p = -\beta(r)$ (or, similarly,$(C1)$ with $\frac{2}{q} = \gamma(r)$) we obtain $(S1')$ through Sobolev embedding.

Now we apply to $(EQ1)$ the corollary of the TT^* lemma with $X_i = L^{\bar{q}}(I, \dot{B}_{\bar{r}}^{\beta(r)})$. We will get

$$||U *_t f||_{L^{q_1}(I, \dot{B}_{r_1}^{-\beta(r_1)})} \leq C||f||_{L^{\bar{q}_2}(I, \dot{B}_{\bar{r}_2}^{\beta(r_2)})}$$

and, again, by posing $p_i = -\beta(r_i)$ we ge $(S2')$ and we can conclude by Sobolev embedding.

Finally, in order to obtain the retarded estimate, by using $(EQ2)$ and by interpolating between $X_0 = L^1(I, L^2)$ and $X_1 = X = L^{\bar{q}}(I, \dot{B}_{\bar{r}}^{\beta(r)})$, we pass through corollary 1.22 and get

$$||U_R * f||_{L^{q_1}(I, \dot{B}_{r_1}^{p_1})} \leq C||f||_{L^{\bar{q}_2}(I, \dot{B}_{r_2}^{-p_2})}$$

where, once again, $p_i = -\beta(r_i)$; now, we arrive at $(S3')$ through Sobolev embedding. $\square$

Chapter 4

Strichartz estimates in L^p spaces and applications

We are now able to translate and rewrite Strichartz estimates in the context of L^p spaces. Here is a version to include the homogeneous case, too:

Theorem 4.1 (Strichartz NLW, homogeneous case)**.** *Let's suppose the (p, q, s) satisfies the following conditions:*

$$(S)\begin{cases} 2 \leq p \leq +\infty \\ 2 \leq q \leq +\infty \\ \frac{1}{p} + \frac{n}{q} = \frac{n}{2} - s \\ \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2} \qquad \text{if } n \leq 3 \\ \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2} \qquad \text{if } n \geq 3 \\ (p, q, s) \neq (2, \infty, 0) \qquad \text{if } n = 3 \end{cases}$$

in this case, given a solution of the wave equation

$$\begin{cases} (-\partial_t^2 + \Delta_{\mathbb{R}^n})u = 0 \\ u(0, x) = g(x) \\ \partial_t u(0, x) = h(x) \end{cases}$$

the following estimate holds:

$$||u||_{L^p([-T,T];L^q(\mathbb{R}^n))} \leq C(||g||_{H^s(\mathbb{R}^n)} + ||h||_{H^{s-1}(\mathbb{R}^n)}).$$

As usual, we denote with p' the conjugated exponent of p, say, $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 4.2 (Strichartz NLW, nonhomogeneous case)**.** *Let's suppose the triplets (p, q, s) and $(r', t', 1-s)$ both satisfy the conditions (S) of the previous theorem.*

In this case, given a solution of the wave equation

$$(*)\begin{cases} (-\partial_t^2 + \Delta_{\mathbb{R}^n})u = F(u) \\ u(0, x) = g(x) \\ \partial_t u(0, x) = h(x) \end{cases}$$

the following estimate holds:

$$||u||_{L^p([-T,T];L^q(\mathbb{R}^n))} \leq C(||g||_{H^s(\mathbb{R}^n)}+||h||_{H^{s-1}(\mathbb{R}^n)}+||F||_{L^r([-T,T];L^t(\mathbb{R}^n))}).$$

These estimates follow directly from their Besov version by embedding operations. Historically speaking, Strichartz created and used them in L^p spaces and provided a proof by using Christ-Kiselev lemma for maximal operators throug a filtration of $\mathbb{R}^n$ and a TT^* argument very similar to the one we presented in Section 1.3.

Let's show, with some examples, the usefulness of these estimates and how, in certain situations, these make the situation better about derivability of inital data in respect of the contraction way of proceeding. From now on, we will set $n = 3$ (tridimensional space).

We observe that if we choose $p = q = 4$ in the (S), we obtain $s = \frac{1}{2}$ and so $(4, 4, \frac{1}{2})$ as admissible triplet. Since $s = 1 - s$ we have $(p, q, s) = (r', t', 1 - s)$ and the dual triplet is made of $(\frac{4}{3}, \frac{4}{3}, \frac{1}{2})$. We get the estimate

$$||u||_{L^4([-T,T];L^4(\mathbb{R}^3))} \leq C(||g||_{H^{\frac{1}{2}}(\mathbb{R}^3)}+||h||_{H^{-\frac{1}{2}}(\mathbb{R}^3)}+||F||_{L^{\frac{4}{3}}([-T,T];L^{\frac{4}{3}}(\mathbb{R}^3))}).$$

Now, we consider a new admissible triplet $(p, q, s) = (\infty, 2, 0)$ (which is not to confuse with the forbidden one, which has the position of p and q exchanged). With some fast computations, $1 - s = 1$ and then $(r', t', 1 - s) = (2, \infty, 1)$, so $(r, t, 1 - s) = (2, 1, 1)$ and finally

$$||u||_{L^\infty([-T,T];L^2(\mathbb{R}^3))} \leq C(||g||_{L^2(\mathbb{R}^3)} + ||h||_{H^{-1}(\mathbb{R}^3)} + ||F||_{L^2([-T,T];L^1(\mathbb{R}^3))}).$$

Now, if one starts from initial data whose norms are very small, we can sum the two estimates

$$||u||_{L^4([-T,T];L^4(\mathbb{R}^3))} + ||u||_{L^\infty([-T,T];L^2(\mathbb{R}^3))} \leq$$

$$\leq C\epsilon + \tilde{C}(||F||_{L^{\frac{4}{3}}([-T,T];L^{\frac{4}{3}}(\mathbb{R}^3))} + ||F||_{L^2([-T,T];L^1(\mathbb{R}^3))}).$$

and one can very often use these estimates in addition with some interpolation methods and some more information about nonlinearity to establish some results about global - not only local - existence of the solution of the wave equation.

We can concretely see one of these situations: we want to show that the problem

$$(*_5) \begin{cases} (-\partial_t^2 + \Delta_{\mathbb{R}^n})u = u^5 \\ u(0,x) = g(x) \\ \partial_t u(0,x) = h(x) \end{cases}$$

admits in $\mathbb{R}^3$ a global solution, that is to say, a solution defined on the entire temporal line $(0, +\infty)$. Moreover we suppose that the inital data (and consequantely u too) have compact support (in respect to the space variables). This interesting result is shown in [9] by using a energy flux estimate and some reasonments which arise from contraption arguments: on the contrary, we will proceed with Strichartz estimates and we will pay the half of the struggle.

Let's introduce two preliminary results

Lemma 4.3. *Let v be a solution of the initial value problem*

$$\begin{cases} (-\partial_t^2 + \Delta_{\mathbb{R}^n})v = f(u) \qquad \text{in } \mathbb{R}^3 \times (0, +\infty) \\ v(0,x) = g(x) \\ \partial_t v(0,x) = h(x) \end{cases}$$

Then $\forall\ T > 0\ \exists\ C = C(T)$ *such that*

$$\sup_{0 \le t \le T} ||v(\cdot,t)||_{L^6(\mathbb{R}^3)} + ||v||_{L^4([0,T];L^{12}(\mathbb{R}^3))} \le C(||Dg||_{L^2(\mathbb{R}^3)} + ||h||_{L^2(\mathbb{R}^3)} + ||f||_{L^1([0,T];L^2(\mathbb{R}^3))})$$

Lemma 4.4 (L^6 estimate)**.** *If u is a weak solution of $(-\partial_t^2 + \Delta_{\mathbb{R}^n})u = u^5$ in $\mathbb{R}^3 \times [0,T)$ then $\forall\ t \in [0,T)$ we have $||u(\cdot,t)||_{L^6(\mathbb{R}^3)} < +\infty$*

Both these facts, which are proved in [26], are a bit technical and not much easy.

We can show what we promised.

We start by showing that $u \in L^4([0,T]; L^{12}(\mathbb{R}^3))$. Let's consider the Strichartz estimate with the admissible triplet $(4, 12, 1)$: we get $1 - s = 0$ and we can choose $(r', t', 1-s) = (\infty, 6, 0)$ as the dual triplet and then $(r, t, 1-s) = (1, \frac{6}{5}, 0)$. We operate a diffeomorphism of the time interval and the inequality reads

$$||u||_{L^4([0,T];L^{12}(\mathbb{R}^3))} \le C(||g||_{H^1(\mathbb{R}^3)} + ||h||_{L^2(\mathbb{R}^3)} + ||u^5||_{L^1([0,T];L^{\frac{6}{5}}(\mathbb{R}^3))}) < \infty$$

not forgetting that $u(\cdot,t) \in L^6(\mathbb{R}^3) \Rightarrow u^5(\cdot,t) \in L^{\frac{6}{5}}(\mathbb{R}^3)$.

Now, we start from the problem $(-\partial_t^2 + \Delta_{\mathbb{R}^n})u - u^5 = 0$ and we differentiate in respect to the space variables, obtaining

$$(-\partial_t^2 + \Delta_{\mathbb{R}^n})v + 5u^4 v = 0$$

where $v = \frac{\partial u}{\partial x_j}$ and $j = 1, 2, 3$.

We apply Lemma 4.3 to this last PDE:

$$\sup_{0 \le t \le \tau} ||Du(\cdot,t)||_{L^6(\mathbb{R}^3)} \le C + C \int_0^\tau ||u^4 Du||_{L^2(\mathbb{R}^3)} dt$$

where the constant C depends of how much small are the initial data and of the fact that $u \in L^4([0,T];L^{12}(\mathbb{R}^3))$, while $0 \leq \tau \leq T$.

We write

$$||u^4||_{L^3} = (\int u^{12})^{\frac{1}{3}} = ((\int u^{12})^{\frac{1}{12}})^4 = ||u||^4_{L^{12}}$$

and so we can apply Hölder inequality

$$||u^4 Du||_{L^2} \leq ||Du||_{L^6}||u^4||_{L^3} = ||Du||_{L^6}||u||^4_{L^{12}}$$

and then take τ small enough such that

$$\int_0^\tau ||u||^4_{L^{12}(\mathbb{R}^3)} dt \leq \frac{1}{2C}$$

we obtain

$$\sup_{0\leq t\leq \tau} ||Du(\cdot,t)||_{L^6(\mathbb{R}^3)} \leq C + C \sup_{0\leq t\leq \tau} ||Du(\cdot,t)||_{L^6(\mathbb{R}^3)} \int_0^\tau ||u^4||_{L^2(\mathbb{R}^3)} dt \leq$$

$$\leq C + \frac{1}{2} \sup_{0\leq t\leq \tau} ||Du(\cdot,t)||_{L^6(\mathbb{R}^3)}$$

that is to say

$$\sup_{0\leq t\leq \tau} ||Du(\cdot,t)||_{L^6(\mathbb{R}^3)} \leq 2C.$$

By repeating this process on every interval of the kind $[k\tau,(k+1)\tau]$, where $k = 0,...,M-1$ e $M\tau = T$, we get

$$\sup_{0\leq t\leq T} ||Du(\cdot,t)||_{L^6(\mathbb{R}^3)} \leq \tilde{C}$$

and so, since has compact support, by Sobolev inequalities,

$$||u||_{L^\infty([0,T]\times\mathbb{R}^3)} \leq ||Du(\cdot,t)||_{L^6(\mathbb{R}^3)} \leq \tilde{C}$$

and u can be prolonged with C^∞-regularity to a global solution of the problem.

Appendix A

Penrose Transform

In recent times some fields, like matemathical physics, suggested different approaches to the study of the NLW, by exploiting more geometrical and metrical properties of the spaces involved. One of these, which has become a classical way of thinking nowadays, is given by the Penrose transform.

Let us start by considering the wave equation in $\mathbb{R}^{n+1}$, equipped with the standard Minkowski metric

$$(*) \begin{cases} \square u = (-\partial_t^2 + \Delta_{\mathbb{R}^n})u = F(t,x) \\ u(0,x) = f_0(x) \\ \partial_t u(0,x) = f_1(x) \end{cases}$$

we are interested in looking for a suitable function space in which we should take the initial data in a way that the global existence of low-range solutions is guaranteed.

The approach, as anticipated, is to compactify the space: this will allow us to look for the existence through a classical contraction argument. In this specific case, the right tool is the use of the Penrose transform:

$$P : \mathbb{R}^{1+n} \to \mathbb{R} \times S^n$$

$$P : (t,x) = (t, |x| \cdot \frac{x}{|x|}) = (t, r\omega) \to (T, R, \omega)$$

defined by

$$(P) \begin{cases} T = \arctan(t+r) + \arctan(t-r) \\ R = \arctan(t+r) - \arctan(t-r) \end{cases}$$

(the angle is not modified by this transform).

It is quite easy to observe that the set $P(\mathbb{R}^{n+1})$ is bounded, due to the fact that $|T| + |R| \leq \pi$: so this is a mapping between the entire Euclidean space and a bounded subset (say, the well-known Penrose-Einstein manifold) of the cylinder $\mathbb{R} \times S^n$.

That means we will able to reconduct the problem $(*)$ to a compact subproblem, on which we'll have more powerful tools to look about the existence of some hypothetical solutions. There is more: we'll prove that the wave operator is sent again in a slightly-modified wave operator, more precisely a costant quantity - which is representing the change of metric between the space and so the change of Gaussian curvature - is added.

Definition A.1. *We call* ***conformal factor*** *of the Penrose Compactification the quantity*

$$\Omega := \cos T + \cos R$$

The name's not casual: the sum of the cosines of the cohordinates inside the Penrose-Einstein manifold is a crucial quantity and, in effect, we're going to show very soon that the mapping between the two metrics is given by a conformal map.

To begin with, we should write an expression of the inverse transormation: by adding and subtracting the two expressions in (P) we get

$$\begin{cases} \frac{T+R}{2} = \arctan(t+r) \\ \frac{T-R}{2} = \arctan(t-r) \end{cases}$$

$$\begin{cases} t+r = \tan(\frac{T+R}{2}) \\ t-r = \tan(\frac{T-R}{2}) \end{cases}$$

$$\begin{cases} 2t = \tan(\frac{T+R}{2}) + \tan(\frac{T-R}{2}) \\ 2r = \tan(\frac{T+R}{2}) - \tan(\frac{T-R}{2}) \end{cases}$$

then we can simplify the righthand terms one last time to get

Lemma A.2.

$$(P^{-1}) \begin{cases} t = \frac{\sin T}{\Omega} \\ r = \frac{\sin R}{\Omega} \\ \omega = \omega \end{cases}$$

We immediately make a computation that will be useful later:

Lemma A.3.

$$\Omega^2 = \frac{4}{(1+(t+r)^2)(1+(t-r)^2)}$$

Proof. Since $t + r = \tan(\frac{T+R}{2})$ we have

$$1 + (t+r)^2 = 1 + \tan^2(\frac{T+R}{2}) = \frac{1}{\cos^2 \frac{T+R}{2}}$$

and in the same way

$$1 + (t-r)^2 = \frac{1}{\cos^2 \frac{T-R}{2}}.$$

By multiplication, we can get

$$(1+(t+r)^2)(1+(t-r)^2) = (\frac{1}{\cos\frac{T+R}{2}\cos\frac{T-R}{2}})^2 =$$

$$= (\frac{2}{\cos(\frac{T+R}{2} + \frac{T-R}{2}) + \cos(\frac{T+R}{2} - \frac{T-R}{2})})^2 =$$

$$= (\frac{2}{\cos T + \cos R})^2 = \frac{4}{\Omega^2}$$

□

Now we're able to prove that the two metrics $\mu = -dt^2 + dx^2$ on $\mathbb{R}^{n+1}$ and $\nu = -dT^2 + d\omega_n^2$ on $\mathbb{R} \times S^{n-1}$ are conformal and, in particular, that the Penrose transform is a conformal map:

Theorem A.4. *By calling $P^*\nu = \nu(T(t,r), R(t,r))$ the pull-back of the metric ν, we have*

$$P^*\nu = \Omega^2 \mu$$

Proof. As

$$\mu = -dt^2 + dx^2 = -dt^2 + dr^2 + r^2 d\omega$$

$$\nu = -dT^2 + d\omega_n^2 = -dT^2 + dR^2 + \sin^2 R d\omega$$

and, remembering the expression of P^{-1}, we have

$$\mu = -dt^2 + dr^2 + r^2 d\omega = -dt^2 + dr^2 + \frac{1}{\Omega^2} \sin^2 R d\omega$$

so, in order to get the aim, it will be enough to show that

$$-dt^2 + dr^2 = \frac{-dT^2 + dR^2}{\Omega^2}.$$

From the expression of (P) we easily get

$$dT = \frac{dt + dr}{1 + (t+r)^2} + \frac{dt - dr}{1 + (t-r)^2}$$

$$dR = \frac{dt + dr}{1 + (t+r)^2} - \frac{dt - dr}{1 + (t-r)^2}$$

and then, with few work, we can simplify the squares and summing the double products:

$$-dT^2 + dR^2 = -4 \frac{dt^2 - dr^2}{(1 + (t+r)^2)(1 + (t-r)^2)}$$

and thanks to the previous lemma we conclude

$$-dT^2 + dR^2 = \Omega^2(-dt^2 + dr^2).$$

□

At this point, we'd like to understand how the wave operator $\Box u$ on $\mathbb{R}^{1+n}$ change after the Penrose transform. We're going to introduce a little bit of notation that, despite being obvious, it's useful not to forget:

$$\Box_\mu \psi = (-\partial_t^2 + \Delta_{\mathbb{R}^n})\psi$$

$$\Box_\nu \psi = (-\partial_T^2 + \Delta_{S^n})\psi$$

The aim will be this one:

Theorem A.5. *Calling $u = \Omega^{\frac{n-1}{2}} v$, it holds*

$$\Box_\mu u = \Omega^{\frac{n+3}{2}} (\Box_\nu - \frac{(n-1)^2}{4}) v$$

As an equivalent fact, we could show the same identity in spherical cohordinates

$$(**) \qquad (-\partial_t^2 + \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{n-1}})u =$$

$$= \Omega^{\frac{n+3}{2}} (-\partial_T^2 + \partial_R^2 + \frac{n-1}{\tan R}\partial_R + \frac{1}{\sin^2 R}\Delta_{S^{n-1}} - \frac{(n-1)^2}{4}) v$$

As a first step, it's necessary to compute some derivatives:

$$u = u(T,R) = u(T(t,r), R(t,r))$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial u}{\partial R}\frac{\partial R}{\partial t}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial T}\frac{\partial T}{\partial r} + \frac{\partial u}{\partial R}\frac{\partial R}{\partial r}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial T^2}(\frac{\partial T}{\partial t})^2 + \frac{\partial^2 u}{\partial T \partial R}\frac{\partial R}{\partial t}\frac{\partial T}{\partial t} + \frac{\partial u}{\partial T}\frac{\partial^2 T}{\partial t^2} + \frac{\partial^2 u}{\partial R \partial T}\frac{\partial T}{\partial t}\frac{\partial R}{\partial t} + \frac{\partial^2 u}{\partial R^2}(\frac{\partial R}{\partial t})^2 + \frac{\partial u}{\partial R}\frac{\partial^2 R}{\partial t^2} =$$

$$= (\frac{\partial T}{\partial t})^2 \frac{\partial^2 u}{\partial T^2} + 2\frac{\partial R}{\partial t}\frac{\partial T}{\partial t}\frac{\partial^2 u}{\partial T \partial R} + (\frac{\partial R}{\partial t})^2\frac{\partial^2 u}{\partial R^2} + \frac{\partial^2 T}{\partial t^2}\frac{\partial u}{\partial T} + \frac{\partial^2 R}{\partial t^2}\frac{\partial u}{\partial R}$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial T^2}(\frac{\partial T}{\partial r})^2 + \frac{\partial^2 u}{\partial T \partial R}\frac{\partial R}{\partial r}\frac{\partial T}{\partial r} + \frac{\partial u}{\partial R}\frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 u}{\partial R \partial T}\frac{\partial T}{\partial r}\frac{\partial R}{\partial r} + \frac{\partial^2 u}{\partial R^2}(\frac{\partial R}{\partial r})^2 + \frac{\partial u}{\partial R}\frac{\partial^2 R}{\partial r^2} =$$

$$= (\frac{\partial T}{\partial r})^2 \frac{\partial^2 u}{\partial T^2} + 2\frac{\partial R}{\partial r}\frac{\partial T}{\partial r}\frac{\partial^2 u}{\partial T \partial R} + (\frac{\partial R}{\partial r})^2\frac{\partial^2 u}{\partial R^2} + \frac{\partial^2 T}{\partial r^2}\frac{\partial u}{\partial T} + \frac{\partial^2 R}{\partial r^2}\frac{\partial u}{\partial R}$$

$$\frac{\partial T}{\partial t} = \frac{1}{1+(t+r)^2} + \frac{1}{1+(t-r)^2} = 2\frac{1+t^2+r^2}{(1+(t+r)^2)(1+(t-r)^2)} = \text{(lemma A.2 and A.3)}$$

$$= \frac{\Omega^2}{2}(1 + \frac{\sin^2 T}{\Omega^2} + \frac{\sin^2 R}{\Omega^2}) = \frac{(\cos T + \cos R)^2 + \sin^2 T + \sin^2 R}{2} = 1 + \cos T \cos R = \frac{\partial R}{\partial r}$$

$$\frac{\partial T}{\partial r} = \frac{1}{1+(t+r)^2} - \frac{1}{1+(t-r)^2} = -4\frac{tr}{(1+(t+r)^2)(1+(t-r)^2)} = \text{(lemma A.2 and A.3)}$$

$$= -\Omega^2 \frac{\sin T}{\Omega}\frac{\sin R}{\Omega} = -\sin T \sin R = \frac{\partial R}{\partial t}$$

With a rapid computation we now show:

Lemma A.6.

$$(-\frac{\partial^2 T}{\partial t^2}+\frac{\partial^2 T}{\partial r^2})=(-\frac{\partial^2 R}{\partial t^2}+\frac{\partial^2 R}{\partial r^2})=0.$$

Proof. The first quantity is 0. in fact:

$$-\frac{\partial^2 T}{\partial t^2}+\frac{\partial^2 T}{\partial r^2}=-\frac{\partial}{\partial t}\frac{\partial T}{\partial t}+\frac{\partial}{\partial r}\frac{\partial T}{\partial r}=-\frac{\partial}{\partial t}\frac{\partial T}{\partial t}+\frac{\partial}{\partial r}\frac{\partial R}{\partial t}=$$

$$-\frac{\partial}{\partial t}\frac{\partial T}{\partial t}+\frac{\partial}{\partial t}\frac{\partial R}{\partial r}=\frac{\partial}{\partial t}(-\frac{\partial T}{\partial t}+\frac{\partial R}{\partial r})=0$$

where we used Schwarz theorem and changed the order of the derivatives (this is fair since the only involved functions are C^2-regular). The second quantity being 0 can be showed with an analogous process. □

Lemma A.7.

$$-(\frac{\partial T}{\partial t})^2+(\frac{\partial T}{\partial r})^2=-\Omega^2;$$

$$-(\frac{\partial R}{\partial t})^2+(\frac{\partial R}{\partial r})^2=\Omega^2;$$

$$-\frac{\partial R}{\partial t}\frac{\partial T}{\partial t}+\frac{\partial R}{\partial r}\frac{\partial T}{\partial r}=0$$

Proof. We begin by showing the second equality is a consequence of the first one and of the fact that

$$\frac{\partial T}{\partial r}=\frac{\partial R}{\partial t},\frac{\partial T}{\partial t}=\frac{\partial R}{\partial r},$$

and the last one is also straightforward.

Coming to the first equality, it's a simple computation:

$$-(\frac{\partial T}{\partial t})^2+(\frac{\partial T}{\partial r})^2=-(1+\cos T\cos R)^2+\sin^2 T\sin^2 R=$$

$$-1-2\cos T\cos R-\cos^2 T\cos^2 R+(1-\cos^2 T)(1-\cos^2 R)=-(\cos T+\cos R)^2=-\Omega^2.$$

□

Now, one last but important consideration:

$$u=\Omega^{\frac{n-1}{2}}v$$

$$\frac{\partial u}{\partial T}=\Omega^{\frac{n-3}{2}}[\Omega\frac{\partial v}{\partial T}-\frac{n-1}{2}\sin Tv]$$

$$\frac{\partial u}{\partial R}=\Omega^{\frac{n-3}{2}}[\Omega\frac{\partial v}{\partial R}-\frac{n-1}{2}\sin Rv]$$

$$\frac{\partial^2 u}{\partial T^2}=\Omega^{\frac{n-5}{2}}[\Omega^2\frac{\partial^2 v}{\partial T^2}-(n-1)\Omega\sin T\frac{\partial v}{\partial T}+(\frac{(n-1)(n-3)}{4}\sin^2 T-\frac{n-1}{2}\Omega\cos T)v]$$

$$\frac{\partial^2 u}{\partial R^2}=\Omega^{\frac{n-5}{2}}[\Omega^2\frac{\partial^2 v}{\partial R^2}-(n-1)\Omega\sin R\frac{\partial v}{\partial R}+(\frac{(n-1)(n-3)}{4}\sin^2 R-\frac{n-1}{2}\Omega\cos R)v]$$

Finally, we can begin to work on the expression of $(**)$. For now, we consider only the lefthand part of the equation:

$$(-\partial_t^2+\partial_r^2+\frac{n-1}{r}\partial_r+\frac{1}{r^2}\Delta_{S^{n-1}})u=$$

$$[-(\frac{\partial T}{\partial t})^2+(\frac{\partial T}{\partial r})^2]\frac{\partial^2 u}{\partial T^2}+2[-\frac{\partial R}{\partial t}\frac{\partial T}{\partial t}+\frac{\partial R}{\partial r}\frac{\partial T}{\partial r}]\frac{\partial^2 u}{\partial T\partial R}+[-(\frac{\partial R}{\partial t})^2+(\frac{\partial R}{\partial r})^2]\frac{\partial^2 u}{\partial R^2}+$$
$$+[-\frac{\partial^2 T}{\partial t^2}+\frac{\partial^2 T}{\partial r^2}]\frac{\partial u}{\partial T}+[-\frac{\partial^2 R}{\partial t^2}+\frac{\partial^2 R}{\partial r^2}]\frac{\partial u}{\partial R}+$$
$$+\frac{n-1}{r}\frac{\partial T}{\partial r}\frac{\partial u}{\partial T}+\frac{n-1}{r}\frac{\partial R}{\partial r}\frac{\partial u}{\partial R}+\frac{1}{r^2}\Delta_{S^{n-1}}u=$$

now, thank to all the rough work done until now,

$$=-\Omega^2\frac{\partial^2 u}{\partial T^2}+\Omega^2\frac{\partial^2 u}{\partial R^2}-\frac{n-1}{r}\sin T\sin R\frac{\partial u}{\partial T}+\frac{n-1}{r}(1+\cos T\cos R)\frac{\partial u}{\partial R}+\frac{1}{r^2}\Delta_{S^{n-1}}u=$$

$$=-\Omega^{\frac{n-1}{2}}[\Omega^2\frac{\partial^2 v}{\partial T^2}-(n-1)\Omega\sin T\frac{\partial v}{\partial T}+(\frac{(n-1)(n-3)}{4}\sin^2 T-\frac{n-1}{2}\Omega\cos T)v]+$$
$$+\Omega^{\frac{n-1}{2}}[\Omega^2\frac{\partial^2 v}{\partial R^2}-(n-1)\Omega\sin R\frac{\partial v}{\partial R}+(\frac{(n-1)(n-3)}{4}\sin^2 R-\frac{n-1}{2}\Omega\cos R)v]-$$
$$-\frac{n-1}{r}\sin T\sin R\Omega^{\frac{n-3}{2}}[\Omega\frac{\partial v}{\partial T}-\frac{n-1}{2}\sin Tv]+$$
$$+\frac{n-1}{r}(1+\cos T\cos R)\Omega^{\frac{n-3}{2}}[\Omega\frac{\partial v}{\partial R}-\frac{n-1}{2}\sin Rv]+$$
$$+\frac{\Omega^{\frac{n-1}{2}}}{r^2}\Delta_{S^{n-1}}v=$$

$$=-\Omega^{\frac{n+3}{2}}\frac{\partial^2 v}{\partial T^2}+\Omega^{\frac{n+3}{2}}\frac{\partial^2 v}{\partial R^2}+\alpha\frac{\partial v}{\partial T}+\beta\frac{\partial v}{\partial R}+\gamma v+\frac{\Omega^{\frac{n+3}{2}}}{\sin^2 R}\Delta_{S^{n-1}}v,$$

where

$$\alpha=(n-1)\sin T\Omega^{\frac{n-1}{2}}(\Omega-\frac{\sin R}{r})=0$$

$$\beta=(n-1)\Omega^{\frac{n-1}{2}}(-\Omega\sin R+\frac{1+\cos T\cos R}{r})=$$
$$=(n-1)\Omega^{\frac{n+1}{2}}\frac{-\sin^2 R+1+\cos T\cos R}{\sin R}=\frac{(n-1)\Omega^{\frac{n+3}{2}}}{\tan R}$$

$$\gamma=\Omega^{\frac{n-1}{2}}[\frac{(n-1)(n-3)}{4}(-\sin^2 T+\sin^2 R)+\frac{n-1}{2}\Omega(\cos T-\cos R)+$$
$$+\frac{(n-1)^2\sin R}{2r\Omega}(\sin^2 T-(1+\cos T\cos R))]=$$
$$=\Omega^{\frac{n+1}{2}}[\frac{(n-1)(n-3)}{4}(\cos T-\cos R)+\frac{n-1}{2}(\cos T-\cos R)-\frac{(n-1)^2}{2}\cos T]=$$
$$=\Omega^{\frac{n+1}{2}}[\frac{(n-1)^2}{4}(\cos T-\cos R)-2\frac{(n-1)^2}{4}\cos T]=$$

$$= \frac{(n-1)^2}{4}\Omega^{\frac{n+1}{2}}[\cos T - \cos R - 2\cos T] = -\frac{(n-1)^2}{4}\Omega^{\frac{n+3}{2}}$$

and we obtain exactly the righthand member of (∗∗), so the theorem is proved.

We write now how the boundary conditions change, but we avoid to repeat all the computations since are very familiar with the ones we already made before:

Lemma A.8. *Si ha*

$$v(0, R\omega) = \Omega^{-\frac{n-1}{2}} f_0(x(0, R\omega))$$

$$\frac{\partial v}{\partial t}(0, R\omega) = \Omega^{-\frac{n-1}{2}}(\frac{\partial t}{\partial T} f_1(x(0, R\omega)) + \frac{\partial r}{\partial T}\sum_{i=1}^{n} \omega_i \frac{\partial f_0(x(0, R\omega))}{\partial x_i})$$

The only open question left is the study of the nonlinearity F (by assuming this is known) and how it changes through the transorm Penrose, since the singularity of the problem is strongly bounded to this very quantity.

Without detailing it too much, we can give the idea of what happens witn a quadratic nonlinearity: we remember that

$$F(t, x) = \Box_\mu u = \Omega^{\frac{n+3}{2}}(\Box_\nu - \frac{(n-1)^2}{4})v$$

$$G(T, X) = \Omega^{-\frac{n+3}{2}} F(t, x) = \Omega^{-\frac{n+3}{2}}(\Omega^{\frac{n-1}{2}})^2 F(T, X) =$$

$$= \Omega^{\frac{n-5}{2}} F[v]$$

We can easily understand that in the case $n \geq 5$ the problem has no singularity (so a Cauchy-Lipschitz argument is enough to guarantee existence and even unicity of a solution). The case $n = 3$ is more delicate: we have a singularity when $\cos(T) = -\cos(R)$, that are exactly the boundary points of the Einstein-Penrose manifold: these singularities can often be bypassed by adding some conditions on the nonlinearity (and they're very often called null conditions). One can read [10] and [21] for more details.

As we stated, using a compactification and small initial data can give information about global existence. One could deduce from the proof of theorem 2.3 that the small data are necessary in this issue:

Theorem A.9. *If in theorem 2.3 we set*

$$E_k(0) = ||g||^2_{H^k(\mathbb{R}^n)} + ||h||^2_{H^{k-1}(\mathbb{R}^n)} < \epsilon$$

theh, when $\epsilon \to 0$, *the life time* T *of the solution lasts not less than* $|\log \epsilon|$.

We finish this section by giving some ideas of the work to do starting from here: Thanks to the local existence theorem, by applying the Penrose transform we deduce the existence and uniqueness v defined on $[-T_\epsilon, T_\epsilon,] \times S^n$. Meanwhile, from theorem A.9, we learn that the smaller ϵ, the bigger the time interval in which the solution is defined. In order to get the goal, however, we only need $T_\epsilon \geq \pi$. In fact, since

$$v \in L^\infty([-T_\epsilon, T_\epsilon], H^k(S^n))$$

one can show that

$$v \in C([-T_\epsilon, T_\epsilon,], H^k(S^n))$$

and a continuous function over a compact set admits maximum and minimum, by Weierstrass theorem: in particular, the function is bounded, say, $|v(T, X)| \leq K$.

Then

$$|u(t,x)| = |\Omega^{\frac{n-1}{2}} v| \leq K \Omega^{\frac{n-1}{2}} \leq \frac{\tilde{K}}{[(1+(t+r)^2)(1+(t-r)^2)]^{\frac{n-1}{4}}}.$$

Again, we leave the details, the proof of the last theorem and the rigorous tractation to [10] and [21].

Appendix B

Sobolev spaces

In this section we will talk a little about some of the definitions, theorems and inequalities used along this work. A more exhaustive and precise composition, with proofs and comments, can be read [4], [5], [6], [9], [15] or [25]; nonetheless, our care drives us to elaborate this knowledge by using the same notations we used before, which very often were different one from the other and so we decided to uniform them under Ginibre-Velo notations as in [12]. In this way, for example, we will denote Sobolev spaces with H^p_r instead of the more common $W^{p,r}$. Or again, we'll use H^p_r instead of $H^p_r(\mathbb{R}^n)$, implying that the construction could be extended to many other manifolds.

It's well known that Sobolev spaces H^p_r where p is integer and positive, are often defined as L^r functions whose derivatives (in distributional sense, like in [4]) are L^r functions. We pose on them the norm

$$||f||_{H^p_r} = \sum_{|\alpha|\leq p} ||\partial^\alpha f||_r$$

where sum is intended over all the multiindices - ordered with any monomial order - whose multidegree is less or equal to p, where obviously $\partial^0 f = f$.

We enumerate some facts:

- H^p_r is a Banach space; H^p_2, often only H^p, is a Hilbert space with the inner product

$$\langle f, g\rangle_{H^p} = \sum_{|\alpha|\leq p} \langle \partial^\alpha f, \partial^\alpha g\rangle_2;$$

- If $f \in H^p_r$ and $|\alpha| \leq p$ then $\partial^\alpha f \in H^{p-|\alpha|}_r$;
- H^p_r is closed for products with Schwartz functions of S, say, $\forall\ \phi \in S, f \in H^p_r, \phi \cdot f \in H^p_r$. Moreover, the Leibniz formula holds;

- The spaces S and C_0^∞ are dense in H_r^p, say, we can approximate every element of H_r^p with a sequence made of Schwartz or compact supported, smooth functions.

We now recall the standard definition of Fourier transform

$$F(f)(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)dx$$

whose inverse is

$$F^{-1}(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} f(\xi)dx.$$

This tool allows us to extend the construction of Sobolev spaces to every possible H_r^p, $1 < p \in \mathbb{R}$ by defining it as the tempered distribution space f such that $F^{-1}((1+|\xi|^2)^{\frac{s}{2}}\hat{f}) \in L^r$, say

$$H_r^p = \{f \in S', F^{-1}((1+|\xi|^2)^{\frac{s}{2}}\hat{f}) \in L^r\}$$

and one can show that

$$||f||_{H_r^p} = ||F^{-1}((1+|\xi|^2)^{\frac{s}{2}}\hat{f})||_{L^r}$$

is compatible (equivalent) with the one we gave with integer exponents. Some authors write $<\xi>=(1+|\xi|^2)$ and call it *Japanese* ξ.

Some more caution is required for homogeneous spaces. Let's come back to integer exponent case: if on the space $\dot{H}_r^p$ we make

$$||f||_{\dot{H}_r^p} = \sum_{|\alpha|=p} ||\partial^\alpha f||_r$$

it's a matter of seconds to realize that this is not a norm, as homogeneous polinomials whose degree is less than p should have norm equal to 0. The right thing to do, this time, is to build the homogeneous space by extracting the elements from S'/P, where P is the polynomial set. In the case of any real exponent, the question is solved by making

$$\dot{H}_r^p = \{f \in S'/P, F^{-1}(|\xi|^p\hat{f}) \in L^r\}$$

(it's useful to notice that Japanese symbol $|\xi|$ is dropped out in order to save rescaling properties). This passage, while sounding authomatical, hides a non trivial issue: nobody guarantees that $|\xi|^p\hat{f}$ is a well-defined distribution, due to the fact that for some exponents $|\xi|^p$ is not smooth. Again, one can bypass this problem by posing

$$\langle|\xi|^p\hat{f}, \phi\rangle = \lim_{\epsilon\to 0}\langle\hat{f}, \mu(\frac{|\xi|}{\epsilon})|\xi|^p\phi\rangle$$

$\forall\ f \in S'/P$, $\phi \in S$ and where $\mu \in C^\infty$ is 1 outside the ball of center 0 and radius 2, while is 0 in the ball of center 0 and radius 1. Now, one defines $\dot{H}^p_r$ as the set of the distributions of S'/P whose limit written before exists, $F^{-1}(|\xi|^p \hat{f})$ exists and is in L^r. It's not surprising that the norm is

$$||f||_{\dot{H}^p_r} = ||F^{-1}(|\xi|^p \hat{f})||_r$$

About these spaces, many other questions arise and, while they deserve some attention and caution, we demand this work to [4] e [6].

We now finist the section with a short view on inequalities we saw along this work, leaving the proofs to [5] or [9].

Theorem B.1 (Gagliardo-Nirenberg-Sobolev)**.** *Let $1 \le p < n$ and $q = p^* = \frac{np}{n-p}$ (such that $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$). Then one can find $\forall\ u \in C^1_0(\mathbb{R}^n)$ a constant $C = C(p,n)$ such that*

$$||u||_q \le C||Du||_p.$$

The compact support request is necessary, as one can see that the inequality fails for $u \equiv 1$.

Theorem B.2 (Poincaré)**.** *Let U a open bounded subset of $\mathbb{R}^n$ and $u \in \dot{H}^1_r(U)$ for some $1 \le r \le n$. Then $\forall\ q$ with $1 \le q \le r^*$ we can find a costant $C = C(q,r,n,U)$ (that is to say, dependent from the open set U too) such that*

$$||u||_{L^q(U)} \le C||Du||_{L^r(U)}.$$

In particular, by choosing $q = r$ (since $1 < r < r^*$), we have

$$||u||_{L^r(U)} \le C||Du||_{L^r(U)}.$$

Theorem B.3 (Sobolev, case H^p_r with $p < \frac{n}{r}$)**.** *Let U be a open bounded subset of $\mathbb{R}^n$. Let $u \in H^p_r$: If $p < \frac{n}{r}$ then $u \in L^q(U)$ with $\frac{1}{q} = \frac{1}{r} - \frac{p}{n}$ and we can find a costant C=C(p,r,n,U) such that*

$$||u||_{L^q(U)} \le C||u||_{H^p_r(U)};$$

Theorem B.4 (Hardy-Littlewood-Sobolev)**.** *Let $n \ge 3$ and $r > 0$. We suppose that $u \in H^1(B(0,r))$. Then $\frac{u}{|x|} \in L^2(B(0,r))$ and it holds*

$$\int_{B(0,r)} (\frac{u}{|x|})^2 dx \le C \int_{B(0,r)} (|Du|^2 + (\frac{u}{r})^2) dr.$$

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